On classification of ruled surfaces

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Preface

The writer worked on ruled surfaces in 1968$\sim$69 and these notes contain the contents of his master thesis and also results obtained through his seminar talks in 1969.

M. Maruyama

January 23, 1970

Contents.

Introduction

Let X be an algebraic curve and consider an exact sequence of algebraic groups over X.

$$(1) \longrightarrow G_m \longrightarrow GL(2) \longrightarrow PGL(1) \longrightarrow (1).$$

This sequence gives an exact sequence of cohomologies

$$H^1(X, G_m) \longrightarrow H^1(X, GL(2)) \longrightarrow H^1(X, PGL(1)) \longrightarrow H^2(X, G_m)$$

here we have $H^2(X, G_m) = 0$ because $\dim X = 1$. This shows that the classification of P^1-bundles*) over X is achieved by classifying vector bundles of rank 2 over X modulo tensor products of line bundles. In Chapter I, we shall study the classification problem of P^1-bundles over X from this view point. Previously, Atiyah [1] studied this problem and his method naturally corresponds to our method in Chapter I ; We shall reproduce in Chapter II the part of the paper [1] which relates to the classification of P^1-bundles.

On the other hand, P^1-bundles over X are naturally regarded as geometric ruled surfaces, and the converse is true (Chapter 0). Nagata [10] gave a complete classification of rational ruled surfaces by a method of elementary transformation, and we studied in [9] a partial classification

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*) In general, we denote by P^r the projective space of $\dim r$.

of geometric ruled surfaces by this method. The contents of [9] and farther consequences will be stated in Chapter III. We shall compare three methods and study special cases in Chapter IV. We can classify a "good half" of P^1-bundles over a complete non-singular curve of arbitrary genus. But another half is very difficult and we can classify it in some special cases.

We shall fix an algebraically closed field k of arbitrary characteristic and consider objects only over k. Let X be a complete non-singular curve of genus g. E, E',... etc. denote always vector bundles of rank 2 over X. I (or, I^2) is the trivial bundle of rank 1 (or, 2, resp.). A subbundle of E always means a sub-linebundle. $\underline{E}$ denotes the sheaf of germs of regular sections of E. k^* denotes the multiplicative group of non-zero elements of k.

Chapter 0. Some preliminary results.

We shall prove, here, some comparison theorems. In the first place,

Theorem 0.1. Let V be a complete variety defined over k. Then, there exists a morphism $\pi : V \longrightarrow X$ such that $\pi^{-1}(x) = P^r$ for all x X if and only if V is a P^r-bundle over X (in Zariski topology).

Proof. The "if" part is obvious. In order to prove the only if part, we first show that a P^r-bundle in étale finite topology is a P^r-bundle in Zariski topology. We consider the following exact sequence of algebraic groups over X :

$$(1) \longrightarrow G_m \longrightarrow GL(r+1) \longrightarrow PGL(r) \longrightarrow (1).$$

Therefore we have an exact sequence of étale finite cohomologies

$$H^1(X_{\text{étf}}, GL(r+1)) \longrightarrow H^1(X_{\text{étf}}, PGL(r)) \longrightarrow H^2(X_{\text{étf}}, G_m),$$

here $H^2(X_{\text{étf}}, G_m) = 0$ because k is algebraically closed and $\dim X = 1$ (see [7]). Thus we know that a P^r-bundle P in the étale finite topology over X is the quotient of a vector bundle E^{r+1} of rank $r + 1$ in the étale finite topology over X by the relation that two points on a line going through the origine of a fibre of E^{r+1} is equivalent. On the other hand, E^{r+1} is locally trivial in the Zariski topology (see [5]). Therefore, P is P^r-bundle in the Zariski topology. Now we go back to the proof of the only if part. By our assumption, π is indeed a proper flat morphism (see Lemma 1.3 of [8]). Therefore V is a P^r-bundle in the étale finite topology over X (see Theorem 8.2 of [6]). Q.E.D.

A complete surface S is called a ruled surface over X if and only if there is a morphism $\pi : S \longrightarrow X$ such that $\pi^{-1}(x) = P^1$ for all $x \in X$. This definition is the same as Nagata's one in [10]. If furthermore X is irrational, then a surface S' which is birational to S is ruled if and only if it is a relatively minimal model.

Now, we obtain directly from the above theorem

Corollary 0.2. A surface S is a ruled surface over X if and only if S is a P^1-bundle over X.

The following theorem shows a relation between the classification of isomorphism classes of P^1-bundles and that of biregular classes of ruled surfaces.

Theorem 0.3. The total spaces of P^1-bundles P, P' over X with positive genus are biregularly equivalent if and only if there is an automorphism ϕ of X such that P is isomorphic to $\phi^*(P')$ *) as P^1-bundle.

Proof. We identify the total space of P^1-bundle P (or, P') with P (or, P', resp.) itself. Let ψ be a biregular map of P to P'. Then

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*) $\phi^*(P)$ is the bundle such that the fibre over $x \in X$ is that of P over $\phi(x)$.

ψ transforms a fibre to a fibre because a rational curve passing through a point of P' is unique by virtue of the assumption that the genus of X is positive. Hence ψ induces a automorphism ϕ of X such that $\phi \circ \pi = \pi' \circ \psi$, where π and π' are projections of P, P' respectively. Thus we have an isomorphism $\psi' : P \longrightarrow \phi^*(P')$ such that the fibre of P over x is transformed to the fibre of P' over x for all $x \in X$. Since automorphisms of P^1 are linear transformations, P is isomorphic to $\phi^*(P')$ as P^1-bundle. Thus we get the "only if" part. The "if" part is obvious. Q.E.D.

The automorphism group of X is a finite group if the genus of X is not less than 2. Hence the classification of biregular classes of ruled surfaces is almost equivalent to that of isomorphism classes of P^1-bundles in that case. But there is a big difference between two classifications if the genus of X is equal to 1. (see Chapter IV § 2 Th.4.7 and Th.4.10)

Chapter I. Vector bundles and P^1-bundles.

In this chapter $L, M, N, \ldots$ etc. denote line bundles over X and $\underline{L}$ does the sheaf of germs of regular sections of L. $L(D)$ denotes the line bundle defined by a divisor D.

§ 1. Some preliminary lemmas

In this section we shall prove some results which will be used later. We start with the following lemma.

Lemma 1.1. Degrees of subbundles of E are bounded above.

Proof. There exist line bundles L_1, L_2 such that there is an exact sequence as follows (see [2])

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0.$$

Let L be an arbitrary subbundle of E. If $L \subset L_1$, then we have $L = L_1$, because $\operatorname{rank} L = \operatorname{rank} L_1 = 1$. Hence $\deg L = \deg L_1$ in this case. Assume that $L \not\subset L_1$. It follows that $\Gamma(X, \underline{L}^{-1} \otimes \underline{L}_2) = \Gamma(X, \underline{\mathrm{Hom}}(L, L_2)) \neq 0$. Hence, $\deg (L^{-1} \otimes L_2) = \deg L_2 - \deg L \geqq 0$, i.e. $\deg L_2 \geqq \deg L$ in this case. Consequently, $\deg L \leqq \max(\deg L_1, \deg L_2)$. Q.E.D.

Definition 1.1. A subbundle L of E is called a maximal subbundle of E if and only if $\deg L$ is maximal. $M(E)$ denotes the maximal degree.

We know that E has at least one subbundle (see [2]). Hence there always exists a maximal subbundle by Lemma 1.1. The following lemma and Corollary 1.6 show that a maximal subbundle of E is uniquely determined under some conditions.

Lemma 1.2. If $\deg E - 2M(E) < 0$, then there is only one maximal subbundle of E.

Proof. Let L_1, L_2 be maximal subbundles of E. We have exact sequences

$$0 \longrightarrow L_1 \xrightarrow{i_1} E \xrightarrow{p_1} L'_1 \longrightarrow 0$$

$$0 \longrightarrow L_2 \xrightarrow{i_2} E \xrightarrow{p_2} L'_2 \longrightarrow 0 .$$

Now, suppose $L_2 \not\subset L_1$, then we obtain $p_1 \circ i_2 \neq 0$. Hence, as in the proof of Lemma 1.1 we see that $\deg L_2 \leqq \deg L'_1$. This cannot occur in our case because $\deg L'_1 = \deg E - \deg L_1 = \deg E - M(E) < M(E) = \deg L_2$. We obtain, therefore, $L_2 \subset L_1$ and they have the same rank, which implies $L_1 = L_2$.

Q.E.D.

Lemma 1.3. If L_1 and L_2 are distinct subbundles of E, then $H^0(X, (\underline{\det E}) \otimes \underline{L}_1^{-1} \otimes \underline{L}_2^{-1})$ is not zero.

Proof. Consider a diagram which consists of two exact sequences

$$\begin{tikzcd} 0 \arrow[r] & L_1 \arrow[r] & E \arrow[r, "h"] & (\det E) \otimes L_1^{-1} \arrow[r] & 0 \\ & & L_2 \arrow[u, "g"'] & & \\ & & 0 \arrow[u] & & \end{tikzcd}$$

Then $g \circ h$ is not zero in $H^0(X, \underline{Hom}(L_2, (\det E) \otimes L_1^{-1})) = H^0(X, (\det E) \otimes L_1^{-1} \otimes L_2^{-1})$. Q.E.D.

The following lemma is important.

Lemma 1.4. If L_1 and L_2 are distinct subbundles of E such that $\deg E = \deg L_1 + \deg L_2$, then we have that $E \cong L_1 \oplus L_2$.

Proof. Consider the sequences as in the proof of Lemma 1.2 for the present L_1 and L_2. We have $(\det E) \otimes L_1^{-1} \otimes L_2^{-1} \cong I$ by Lemma 1.3 and our assumption. Hence we get an isomorphism $h : (\det E) \otimes L_1^{-1} \longrightarrow L_2$. Now, the homomorphism $h \circ p_1 \circ i_2 \neq 0$ is an element of $\Gamma(X, \underline{Hom}(L_2, L_2)) = k$. Thus there is an element α in $k - \{0\}$ such that $h \circ p_1 \circ i_2$ is the multiplication by α. Since $(\alpha^{-1} \circ h \circ p_1) \circ i_2 = \mathrm{id}$, the sequence $0 \longrightarrow L_2 \longrightarrow E \longrightarrow L_2' \longrightarrow 0$ splits. Accordingly, we obtain that E is isomorphic to $L_1 \oplus L_2$. Q.E.D.

Maximal subbundles of E cannot be isomorphic each other except for some special cases. In fact, we have

Lemma 1.5. If L_1 and L_2 are distinct maximal subbundles of E and $L_1 \cong L_2$, then $E = L_1 \oplus L_1$.

Proof. Let $E' = E \otimes L_1^{-1}$. Then the trivial bundles $L_1 \otimes L_1^{-1}$ and $L_2 \otimes L_1^{-1}$ are maximal subbundles of E'. Thus we have $\dim H^0(X, E') \geqq 2$. On the other hand, the global sections which are not the zero-section have no zero-points. For, if a section $\phi \in H^0(X, E')$ have zero-points, $[\phi]$

is a subbundle of E' with positive degree (see [2] p.419). Since E' is of rank 2, one can see that E' is isomorphic to the trivial bundle. Hence we get $E \cong L_1 \oplus L_1$. Q.E.D.

Corollary 1.6. (i) If $\deg E - 2\,M(E) = 0$ and if E is indecomposable, then a maximal subbundle of E is unique.

(ii) If $\deg E - 2\,M(E) = 0$, E is decomposable and if $E \not\cong L \oplus L$ for any subbundle L of E, then there are only two maximal subbundles of E.

Proof. Let L_1, L_2 be distinct maximal subbundles. Then, since $\deg E - \deg L_1 - \deg L_2 = 0$, we have $E = L_1 \oplus L_2$ by Lemma 1.4, which proves (i). As for (ii), the hypotheses imply not only that $E = M_1 \oplus M_2$ $(M_1 \not\cong M_2)$ but also that M_1 and M_2 are maximal subbundles of E. Let L be a maximal subbundle of E. Then the exact sequence

$$0 \longrightarrow M_1 \longrightarrow E \longrightarrow M_2 \longrightarrow 0$$

and a similar argument as in the proof of Lemma 1.2 show that $L = M_1$ or $L \cong M_2$. If $L \cong M_2$, we see that $L = M_2$ by Lemma 1.5. Q.E.D.

Remark 1.7. It is clear that it holds that if $E = L \oplus L$, then E has infinitely many maximal subbundles. But all maximal subbundles are isomorphic to L in the case.

The integer $\deg E - 2\,M(E)$ is bounded above when E ranges over all vector bundles of rank 2 over X. In fact, we have

Lemma 1.8.*)

$$\deg E - 2\, M(E) \leqq \begin{cases} 2g - 1 & \text{if} \quad g \geqq 1 \\ 0 & \text{if} \quad g = 0 \end{cases} \qquad \text{for all} \quad E.$$

Proof. Let L be a maximal subbundle of E and set $E' = E \otimes L^{-1}$. Then we obtain an exact sequence

$$0 \longrightarrow I \longrightarrow E' \longrightarrow L' \longrightarrow 0$$

and $\deg E' = \deg L' = \deg E - 2\, M(E)$. Now, suppose that $\deg E - 2\, M(E) \geqq 2g$. then the Riemann-Roch theorem over X implies that $\dim H^0(X, \underline{E}') \geqq \deg E' + 2(1 - g) \geqq 2$ (see [2] or [4]). However, I is maximal subbundle of E', because L is such one of E. Thus one can see that E' is the trivial bundle by the same argument as in the proof of Lemma 1.5. Hence, $\deg E - 2\, M(E) = \deg E' = 0$, which is a contradiction if $g > 0$. In case where $g = 0$, by the same argument we see that $\deg E - 2\, M(E) > 0$ implies that E' is trivial and $\dim H^0(X, \underline{E}') \geqq 3$, which is impossible. Q.E.D.

§ 2. Invariants and classification

From now on, a vector bundle (of rank 1 or 2) is identified with

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*) The lowest upper bound of $\deg E - 2\, M(E)$ is g. This will be proved in Chapter III, § 1.

another bundle if they are isomorphic to each other. Let ε_X be the set of the isomorphism classes of vector bundles of rank 2 over X. We introduce the following relation in ε_X.

Definition 1.2. $E_1, E_2 \in \varepsilon_X$ are called equivalent if and only if there exists a line bundle L such that $E_1 = E_2 \otimes L$. Then we denote this relation by $E_1 \sim E_2$.

It is obvious that the relation $\sim$ is an equivalence relation. Let $\mathcal{P}_X$ be the quotient set $\mathcal{E}_X/\sim$. $\mathcal{P}_X$ can be identified with the set of isomorphism classes of P^1-bundles over X (see Introduction). The class of E in $\mathcal{P}_X$ is denoted by P(E) and P(E) is regarded as a P^1-bundle too.

Now, put $N(E) = \deg E - 2\,M(E)$ and $\mathcal{A}(E) = \{\det E \otimes L^{-2}\}$, where L ranges over all maximal subbundles of E and if $L_1^{-2} \cong L_2^{-2} \cong, \ldots, \cong L_r^{-2}$ for r maximal subbundles $L_1, L_2, \ldots, L_r$, $\det E \otimes L_1^{-2}$ is counted r times. The degrees of elements of $\mathcal{A}(E)$ are N(E). N(E) and $\mathcal{A}(E)$ satisfy the following proposition.

Proposition 1.9.

(i) N(E) is an integer and is not greater than g.

(ii) Both N(E) and $\mathcal{A}(E)$ depend only on P(E)

(iii) $\mathcal{A}(E)$ contains only one element if one of the following conditions is satisfied.

(a) $N(E) < 0$

(b) $N(E) = 0$ and E is indecomposable.

(iv) $\mathcal{S}(E)$ contains only two elements and they are dual each other if $N(E) = 0$, E is decomposable and $P(E) \neq P(I^2)$.

Proof. It is obvious that $N(E)$ is an integer. The fact that $N(E)$ is not greater than g will be proved in Chapter III, § 1. (ii) is clear if one notes that M is a maximal subbundle of E if and only if $M \otimes L$ is a maximal subbundle of $E \otimes L$. As for (iii), (a) is an immediate consequence of Lemma 1.2 and (b) follows from Corollary 1.6, (i). (iv) follows from Corollary 1.6, (ii). Q.E.D.

The following notion is now adequate.

Definition 1.3. A vector bundle E of rank 2 is called of canonical type if and only if I is a maximal subbundle of E.

It is clear that $P(E)$ contains at least one vector bundle of canonical type. Thus, if $P(E)$ has only one vector bundle of canonical type, the classification of $\mathcal{V}_X$ is reduced to that of vector bundles of canonical type. In fact, under a certain condition $P(E)$ determines uniquely a vector bundle of canonical type. But the determination is not always true (see Chapter IV § 2).

Lemma 1.10.

(i) Under one of the conditions of Proposition 1.9, (iii), $P(E)$

contains only one vector bundle of canonical type.

(ii) If $E = L_1 \oplus L_2$, $N(E) = 0$, (hence $\deg L_1 = \deg L_2$) and $L_1 \not\cong L_2$, then the vector bundles of canonical type in $P(E)$ are $I \oplus (L_2 \otimes L_1^{-1})$ and $(L_1 \otimes L_2^{-1}) \oplus I$.

Proof. All our assertions follow from the facts which were used in the proof of Proposition 1.9.

Put $C_X^0 = \{(D, \xi) \mid D =$ divisor class on X with $\deg D \leqq 0$, $\xi \in P(H^1(X, \underline{L(-D)})) \cup \{0\}\}$, where $P(H^1(X, \underline{L(-D)}))$ is the projective space $H^1(X, \underline{L(-D)}) - \{0\}/k^*$. Let C_X be the quotient set of C_X^0 by the relation such that (D, ξ) and (D', ξ') are equivalent if and only if (i) $D = D'$ and $\xi = \xi'$, or (ii) $D' = -D$ and $\xi = \xi' = 0$. Then we get the following theorem.

Theorem 1.11. $\mathcal{P}_X^- = \{P(E) \mid P(E) \in \mathcal{P}_X$ and $N(P(E)) \leqq 0\}$ bijectively corresponds to C_X.

Proof. Let $P(E)$ be an element of $\mathcal{P}_X^-$ and E' be a vector bundle of canonical type in $P(E)$. Then E' an extension of $\det E'$ by I and $\det E' \in \mathcal{S}(P(E))$:

$$0 \longrightarrow I \xrightarrow{i} E' \xrightarrow{p} \det E' \longrightarrow 0.$$

To give an isomorphism class of non-trivial extensions is equivalent to give an element of $P(H^1(X, \underline{(\det E')^{-1}}))$ because the injection i is uniquely determined up to multiplication of elements of k^* by Lemma 1.5 (see [3]). Therefore, extension classes of $\det E'$ by I and $P(H^1(X, \det E')^{-1})) \cup \{0\}$ are bijective, the trivial extension corresponding to $\{0\}$. Thus by Proposition 1.9 and Lemma 1.10 $P(E)$ determines an element of C_X since an isomorphism class of line bundles defines a divisor class. Conversely, an element (D, ξ) of C_X^0 gives an extension class.

$$0 \longrightarrow I \longrightarrow E(D, \xi) \longrightarrow L(D) \longrightarrow 0.$$

Since $\deg L(D) \leqq 0$, I is a maximal subbundle of $E(D, \xi)$ (see the proof of Lemma 1.1) and so $E(D, \xi)$ is of canonical type. $P(E(D, \xi))$ is, therefore, an element of $\mathcal{P}_X^-$ and if (D, ξ) and (D', ξ') are mapped to the same element of C_X by the canonical map, then $P(E(D, \xi)) = P(E(D', \xi'))$ It is easy to see that two maps defined as above are reciprocal to each other.

Q.E.D.

When $P(E) \in \mathcal{P}_X^-$ corresponds to the class of (D, ξ) in C_X by the above correspondence, $L(D)$ is an element of $\mathcal{S}(P(E))$. On the other hand, $\mathcal{S}(P(E))$ has one or two elements. Hence we shall use $\mathcal{S}(P(E))$ instead of D or $L(D)$, and identify the class of (D, ξ) in C_X with $(\mathcal{S}(P(E)), \xi(P(E)))$.

Corollary 1.12. The moduli space of P^1-bundles over X of genus $g \geqq 1$ which are decomposable, that is, P^1-bundle $P(E)$ such that E is decomposable, is union of J_n (n ranges over all negative integers) and $\tilde{J}_0$, where J_n is biregularly isomorphic to the Jacobian variety of X and $\tilde{J}_0$ is the quotient space of the Jacobian variety by the relation $A \equiv -A$.

Proof. Decomposable P^1-bundles are contained in $\mathfrak{P}_X^-$ because a direct summand of a decomposable bundle whose degree is not less than that of another summand is a maximal subbundle of the bundle (see the proof of Lemma 1.1). Moreover, a P^1-bundle $P(E)$ is decomposable if and only if the corresponding element in C_X is $(\mathfrak{d}(P(E)), 0)$. $\{(\mathfrak{d}(P(E)), 0) \mid N(P(E)) = n < 0\}$ and $\{(\mathfrak{d}(P(E)), 0) \mid N(P(E)) = 0\}$ are in bijective correspondence with J_n and $\tilde{J}_0$ respectively. Q.E.D.

Remark 1.13. If $N(P(E)) < 2 - 2g$, then $H^1(X, \mathfrak{d}(P(E))^{-1}) = 0$, that is to say, $P(E)$ is decomposable. Therefore, by the above corollary it is important to classify $P(E)$ of $\mathfrak{P}_X$ such that $2 - 2g \leqq N(P(E)) \leqq g$. But it is very difficult to classify $\mathfrak{P}_X^+ = \{P(E) \mid N(P(E)) > 0\}$ (see Chapter III, 1 and Chapter IV, § 2).

Remark 1.14. det E, the maximal subbundle of E and $\xi(P(E))$ classify the vector bundles of rank 2 such that $N(E) \leqq 0$.

§ 3. Geometric meaning of invariants.

In the preceding section, we have defined the invariants $N(P(E))$ and

(P(E)). We shall study, here, the geometric meaning of them. We shall begin with

Lemma 1.14. To give a section of $P(E)$ is equivalent to give a subbundle of E.

Proof. Let $\{U_i\}$ be a system of sufficiently fine coordinate neighbourhood of E and let L be a subbundle of E. If the transition matrices of L are (a_{ij}), then those of E are of the form $\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & c_{ij} \end{pmatrix}$ in the suitable coordinates. When the coordinates of E are $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$, those of $P(E)$ are $z_i = x_i/y_i$ in a natural correspondence and the coordinate transformations are $z_i = (a_{ij}/c_{ij})z_j + (b_{ij}/c_{ij})$. Then, the subbundle L corresponds to the infinite section (i.e. the section defined by $z_i^{-1} = 0$) of $P(E)$ in the above coordinates because L is defined by $y_i = 0$ in E. This defines a map Υ of the set of subbundles to that of sections. A subbundle different from L cannot correspond the infinite section with respect to the above coordinates. Thus the map Υ is injective. Conversely, let s be a section of $P(E)$. If one takes s as the infinite section, the coordinate transformations are $z_i = d_{ij} \cdot z_j + f_{ij}$. The transition matrices of E are $\begin{pmatrix} a'_{ij} & b'_{ij} \\ 0 & c'_{ij} \end{pmatrix}$ with respect to the corresponding coordinates of E, where $a'_{ij}/c'_{ij} = d_{ij}$ and $b'_{ij}/c'_{ij} = f_{ij}$.

Then the subset of E such that $y_i = 0$ is the subbundle L' of E defined by the transition matrices (a'_{ij}). Now, L' corresponds to s by γ, that is, γ is surjective. Q.E.D.

$N(P(E))$ has the following meaning.

Lemma 1.15. $N(P(E))$ is the minimum of self-intersection numbers of sections of $P(E)$.

Proof. Let s be a section of $P(E)$ and L_s be the corresponding subbundle of E. Then we obtain an exact sequence

$$0 \longrightarrow L_s \longrightarrow E \longrightarrow L \longrightarrow 0.$$

We can take a coordinate of E such that the transition matrices are of the form $\begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix}$, (a_{ij}) and (b_{ij}) defining L_s and L respectively. Then, we may assume that $P(E)$ has the coordinates such that the coordinate transformations are $z_i = (a_{ij}/b_{ij})z_j + (c_{ij}/b_{ij})$ and s is the infinite section. When x is a local coordinate of X, the local coordinates of the ruled surface $P(E)$ in a neighbourhood of s may be taken as (x, z_i^{-1}). The local equations of s are given by $z_i^{-1} = 0$ and so the characteristic bundle of s are given by transition matrices

$$g_{ij} = (z_i^{-1}/z_j^{-1})_s = (z_i^{-1}(\frac{a_{ji}}{b_{ji}} z_i + \frac{c_{ji}}{b_{ji}}))_s = a_{ji}/b_{ji} = b_{ij}/a_{ij}.$$

Therefore, if $d(L \otimes L_s^{-1})$ denotes the divisor class defined by $L \otimes L_s^{-1}$, we have $\pi(s \cdot s)$*) $= d(L \otimes L_s^{-1})$, where π is the projection of $P(E)$ onto X. On the other hand, $\deg(L \otimes L_s^{-1})$ is minimal if and only if L_s is a maximal subbundle of E, and if L_s is such a subbundle, we obtain (s,s)**) $= \deg(L \otimes L_s^{-1}) = N(P(E))$. Q.E.D.

We employ the following definition.

Definition 1.4. A section s of $P(E)$ is called a minimal section of $P(E)$ if and only if $(s, s) = N(P(E))$.

Lemma 1.14 and Lemma 1.15 imply that the minimal sections of $P(E)$ are in bijective correspondence with the maximal subbundles of E. Moreover, the proof of Lemma 1.15 above shows that if s is a minimal section of $P(E)$, then $\pi(s \cdot s) = d\,((\det E) \otimes L_s^{-2})$ for the corresponding maximal subbundle L_s of E. On the other hand, the map : $L \longrightarrow (\det E) \otimes L^{-2}$ of the set of maximal subbundles into $\mathcal{S}(P(E))$ is bijective by the definition of $\mathcal{S}\,(P(E))$. Thus we have

Theorem 1.16. The set of minimal sections of $P(E)$ is bijective with the set of maximal subbundles of E. Moreover, if s is a minimal section of $P(E)$, then $L(\pi(s \cdot s))$ (π is the projection of $P(E)$ onto X) is an element of $\mathcal{S}\,(P(E))$ and the map : $s \longrightarrow L(\pi(s \cdot s))$ of the set of

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*) $s \cdot s'$ is the intersection of s and s'.

**) (s,s) is the self-intersection number of s.

minimal sections of $P(E)$ into $\mathfrak{S}(P(E))$ is bijective.

Combining Proposition 1.9 and the above theorem, we get

Corollary 1.17. If $N(P(E)) < 0$ or if $N(P(E)) = 0$ and E is indecomposable, then $P(E)$ has only one minimal section. On the other hand, if $N(P(E)) = 0$, E is decomposable and if $P(E)$ is not the trivial bundle, then $P(E)$ has only two minimal sections.

For the self-intersection number of an arbitrary section, we have the following result.

Proposition 1.18. Let s be a section of $P(E)$ which is not a minimal section.

(i) If $N(P(E)) < 0$, then $(s, s) \geqq -N(P(E))$.

(ii) If $N(P(E)) \geqq 0$, then $(s,s) \geqq 2 + N(P(E))$.

Moreover, if $N(P(E))$ is even, then (s,s) is even and if $N(P(E))$ is odd, then (s,s) is odd.

Proof. Let s' be a minimal section and let L_s and $L_{s'}$ be the subbundles of E corresponding to s and s' respectively. By virtue of the proof of Lemma 1.15, $(s,s) = \deg((\det E) \otimes L_s^{-2})$ and $(s',s') = \deg((\det E) \otimes L_{s'}^{-2})$. Hence, $(s,s) + (s',s') = 2\deg((\det E) \otimes L_s^{-1} \otimes L_{s'}^{-1})$. On the other hand, $\deg((\det E) \otimes L_s^{-1} \otimes L_{s'}^{-1}) \geqq 0$ by Lemma 1.3. Thus, $(s,s) + (s',s') = 2r$ $(r \geqq 0)$ and this proves all our assertions. Q.E.D.

Remark 1.19. Let s and s' be distinct sections of $P(E)$ and let L_s and $L_{s'}$ be subbundles of E corresponding to s and s' respectively. Then we have that $\pi(s\cdot s') \in |(\det E)\otimes L_s^{-1}\otimes L_s^{-1}|$, where π is the projection morphism of $P(E)$ and $|L|$ is the complete linear system of a divisor defined by L. Thus, $\pi(s\cdot s) + \pi(s'\cdot s') = 2\pi(s\cdot s')$, if one regards $\pi(s\cdot s)$, $\pi(s'\cdot s')$ and $\pi(s'\cdot s')$ as the divisor classes on X.

Proof. Let (a_{ij}), (b_{ij}) and (d_{ij}) be the transition matrices of line bundles L_s, $(\det E)\otimes L_s^{-1}$ and $L_{s'}$ respectively. Then we may assume that the transition matrices of E are $\begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix}$. Since $L_{s'}$ is a subbundle of E, there exists a set $\{(f_i, g_i)\}$, where f_i, g_i are regular functions on the coordinate neighbourhood U_i such that f_i and g_i are not simultaneously zero at any point and that $\begin{pmatrix} f_i \\ g_i \end{pmatrix} d_{ij} = \begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix}\begin{pmatrix} f_j \\ g_j \end{pmatrix}$. Hence we obtain the relation $g_i = (b_{ij}/d_{ij})g_j$, that is, $\{g_i\}$ is a regular section of $(\det E)\otimes L_s^{-1}\otimes L_{s'}^{-1}$. On the other hand, s intersects s' at the zero-points of $\{g_i\}$ with the same mulplicity as the order of zero of $\{g_i\}$. And the divisor of the zeros of the regular section $\{g_i\}$ is contained in $|(\det E)\otimes L_s^{-1}\otimes L_{s'}^{-1}|$. This proves the first half of our assertions. The latter half is easy if one notes that $\pi(s\cdot s) + \pi(s'\cdot s') = 2\cdot d(\det E\otimes L_s^{-1}\otimes L_{s'}^{-1})$ (d is the same as in the proof of Lemma 1.15). Q.E.D.

Remark 1.20. We know by the above remark that Lemma 1.4 means the following fact ; $P(E)$ is decomposable if and only if there exist two sections in $P(E)$ which do not intersect each other.

Chapter II. Atiyah's method.

Let A be a group of affine transformations. An A-bundle is a fibre bundle such that the fibre of it is P^1 and the transformation group is A, that is to say, $z \longrightarrow az + b$, $z \in P^1$. Atiyah classified A-bundles over an arbitrary complete non-singular variety. And he showed that a P^1-bundle (whose transformation group is $PGL(1)$) over a complete non-singular curve is represented by an A-bundle and studied a criterion for two A-bundles to be isomorphic as P^1-bundles. Atiyah assumed that k is the complex number field, but this assumption is not necessary for the classification of P^1-bundles. Therefore we suppose that k is an algebraically closed field of arbitrary characteristic in this chapter too, and we reproduce the contents of Atiyah's paper [1] for the convenience of readers.

§ 1. A-bundles.

In this section, V is a complete non-singular variety of arbitrary dimension over the field k. Consider the following exact sequence of algebraic groups over V :

$$0 \longrightarrow G_a \longrightarrow A \longrightarrow G_m \longrightarrow 0.$$

Then we have a sequence of cohomologies

$$H^1(V, G_a) \longrightarrow H^1(V, A) \longrightarrow H^1(V, G_m).$$

Since $H^1(V, G_m)$ is in bijective correspondence with the divisor class group of V, we can classify A-bundles by classifying $\gamma^{-1}(D)$, where D ranges over divisor classes of V. Let $\alpha = \{U_i\}$ be an open covering of V. We shall consider, first, the classification of A-bundles with coordinate neighbourhoods α and secondly, classify all A-bundles by taking inductive limits. Let γ_α be the natural map : $H^1(\alpha, A) \longrightarrow H^1(\alpha, G_m)$ and let η, $\eta' \in H^1(\alpha, A)$. Then η (or, η', respectively) is defined by the coordinate transformations $z_i = a_{ij}z_j + b_{ij}$ (or, $z_i = a'_{ij}z_j + b'_{ij}$, respectively), where $a_{ij} \neq 0$ (or, $a'_{ij} \neq 0$, resp.) and a_{ij}, a'_{ij}, b_{ij}, b'_{ij} are regular functions on $U_i \cap U_j$. If η and η' have the same image by the map γ_α, then there are invertible regular functions ϕ_i on U_i such that $a_{ij} = \phi_i^{-1}a'_{ij}\psi_j$ in $U_i \cap U_j$. Then η' is defined by the coordinate transformations $z_i = a_{ij}z_j + \phi_i^{-1}b'_{ij}$. Hence every element of $\gamma_\alpha^{-1}(\xi)$ is represented by $\{a_{ij}, b_{ij}\}$ with fixed $\{a_{ij}\}$, where ξ is the G_m-bundle defined by $\{a_{ij}\}$. Now, $\{a_{ij}, b_{ij}\}$ and $\{a_{ij}, b'_{ij}\}$ represent the same element of $\gamma_\alpha^{-1}(\xi)$ if and only if there exist regular functions ϕ_i, ψ_i on U_i such that $\begin{pmatrix} a_{ij} & b'_{ij} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \phi_i & \psi_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_j & \psi_j \\ 0 & 1 \end{pmatrix}^{-1}$ in $U_i \cap U_j$. This relation is equivalent to the following :

$$a_{ij} = \phi_i a_{ij} \phi_j^{-1}$$

$$a_{ij}\psi_j + b'_{ij} = \phi_i b_{ij} + \psi_i$$

The former implies that $\phi_i = \phi_j$ in $U_i \cap U_j$, whence ϕ_i is a regular function over V, i.e. ϕ_i is a constant $r \in k^*$. Therefore, the latter relation reduces to the following :

$$b'_{ij} - rb_{ij} = \psi_i - a_{ij}\psi_j \qquad \text{in } U_i \cap U_j \qquad (1).$$

On the other hand, Since $\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix}$ defines an A-bundle, we obtain that $\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{jk} & b_{jk} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{ki} & b_{ki} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $U_i \cap U_j \cap U_k$, and $\begin{pmatrix} a_{ii} & b_{ii} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in U_i, that is,

$$\left.\begin{array}{ll} a_{ij}a_{jk}b_{ki} + a_{ij}b_{jk} + b_{ij} = 0 & \text{in } U_i \cap U_j \cap U_k \\ b_{ii} = 0 & \text{in } U_i \end{array}\right. \qquad (2).$$

Since $\{a_{ij}\}$ defines a divisor, there are functions h_i on U_i such that $a_{ij} = h_i/h_j$ in $U_i \cap U_j$. If we put $c_{ij} = b_{ij}/h_i$, $c'_{ij} = b'_{ij}/h_i$, then the equations (1) and (2) are equivalent to the following :

$$c'_{ij} - rc_{ij} = \theta_i - \theta_j \qquad \text{in } U_i \cap U_j \tag{3}$$

$$\begin{aligned} c_{ij} + c_{jk} + c_{ki} &= 0 \qquad \text{in } U_i \cap U_j \cap U_k \\ c_{ii} &= 0 \qquad \text{in } U_i \end{aligned} \tag{4}$$

where $\theta_i = \psi_i/h_i$. Since b_{ij} is a regular function on $U_i \cap U_j$ and h_i is a local parameter of a divisor D, $\{c_{ij}\}$ defines an element of $H^1(\alpha, \Omega(D))$ by virtue of (4), where $\Omega(D)$ is the sheaf of germs of functions f such that locally $(f) > -D$. Thus we get a surjective map $\Phi_\alpha : H^1(\alpha, \Omega(D)) \longrightarrow \gamma_\alpha^{-1}(\xi)$. The equation (3) implies that $\Phi_\alpha(u) = \Phi_\alpha(u')$ if and only if $u = ru'$ for an element r of k^*. Taking inductive limits, we obtain a surjective map $\Phi : H^1(V, \Omega(D)) \longrightarrow \gamma^{-1}(D)$ and $\Phi(u) = \Phi(u')$ for $u, u' \in H^1(X, \Omega(D))$ if and only if $u = ru'$ for an element r of k^*. In paticular $\eta \in \gamma^{-1}(D)$ is the G_m-bundle defined by D if and only if $\eta = \Phi(0)$. If one notes that the sheaf $\Omega(D)$ is isomorphic to $\underline{L(D)}$, our result is formulated as follows :

Theorem 2.1. The set of A-bundles giving rise to a given G_m-bundle $\xi \in H^1(V, G_m)$ consists of ξ itself and the projective space $P(H^1(V, \underline{L(D)}))$, where D is the divisor class defined by ξ.

If $\gamma(\eta) = D$, then η is defined by the coordinate transformations $\{z_i = a_{ij}z_j + b_{ij}\}$, where D is determined by $\{a_{ij}\}$. Then

$$\pi(s_\infty \cdot s_\infty) = -D \tag{5}$$

as in the proof of Lemma 1.15 when s_∞ is the infinite section of η and π is the natural projection of η .

§ 2. Geometry of A-bundles.

Now, we shall assume again that X is a complete non-singular curve over the field k. Let $P(E)$ be a P^1-bundle over X. Since E has at least one subbundle, the transition matrices of E are of the form $\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & c_{ij} \end{pmatrix}$ under a suitable coordinate system. Thus $P(E)$ is represented by an A-bundle.

If C is a positive divisor on X, then we get an injective morphism : $\Omega(D) \longrightarrow \Omega(D + C)$, which gives a pair of morphisms β and β^*, which are dual to each other by the Serre duality , as follows :

$$\begin{array}{c} H^1(X, \Omega(D)) \xrightarrow{\beta} H^1(X, \Omega(D + C)) \\ \\ H^0(X, \Omega(K - D)) \xleftarrow{\beta^*} H^0(X, \Omega(K - D - C)). \end{array} \tag{6}$$

Since β^* is injective, β is surjective. If $\deg C$ is sufficiently large, $H^0(X, \Omega(K - D - C)) = 0$, hence $H^1(X, \Omega(D + C)) = 0$. Thus there is a divisor C on X such that $\beta(\zeta) = 0$ for all $\zeta \in H^1(X, \Omega(D))$. So

ζ is represented by a cocycle $\{c_{ij}\}$ such that

$$c_{ij} = \frac{f_i}{h_i} - \frac{f_j}{h_j} \qquad \text{in } U_i \cap U_j \tag{7}$$

where $(f_i) > -C$ in U_i and $\{h_i\}$ are the local parameters of D. If $c'_{ij} = ((f_i + g_i)/h_i) - ((f_j + g_j)/h_j)$ and if each g_i is a regular function on U_i, then $\{c_{ij}\}$ and $\{c'_{ij}\}$ define the same element in $H^1(X, \Omega(D))$. Therefore, ζ depends only on the principal part*) of $\{f_i\}$, which we shall denote by ϕ. Conversely, for a given principal part ϕ, we can choose a sufficiently fine covering $\{U_i\}$ and rational functions $\{f_i\}$ such that f_i defines ϕ and it has no pole in $U_i \cap U_j$ if $i \neq j$. Then $\{c_{ij}\}$, given by the equation (7), determines an element of $H^1(X, \Omega(D))$. It is clear that the principal part $r_1\phi_1 + r_2\phi_2$ corresponds to $r_1\zeta_1 + r_2\zeta_2$ if each ϕ_i corresponds to ζ_i and $r_1, r_2 \in k$. Hence we have a surjective linear map : $\mathcal{L} \longrightarrow H^1(X, \Omega(D))$, where $\mathcal{L}$ is the vector space of all principal parts over X. On the other hand, $\zeta \in H^1(X, \Omega(D))$ defines an element $\eta \in \gamma^{-1}(D)$ by Theorem 2.1, given by $\{a_{ij}, b_{ij}\}$, where

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*) We can choose a sufficiently fine covering $\{U_i\}$ such that has no pole in $U_i \cap U_j$ $i \neq j$. For any point $Q \in X$ fix a uniformanizing parameter $t(Q)$ at Q. If f_i has a pole of order n at Q U_i, then $f_i = a_n t(Q)^{-n} + \ldots + a_1 t(Q)^{-1} + b$, where b is regular at Q. Then $a_n t(Q)^{-n} + \ldots + a_1 t(Q)^{-1}$ is the principal part of f_i at Q.

$b_{ij} = h_i c_{ij}$. Then, from the equation (7), we have

$$b_{ij} = f_i - a_{ij} f_j \qquad \text{in } U_i \cap U_j ,$$

which shows that $\{f_i\}$ defines a section of the bundle space of

Therefore, we have a surjection $\sigma : \mathcal{L} \longrightarrow \gamma^{-1}(D)$. If $\eta = \sigma(\phi)$, then η has a section with principal part ϕ, and conversely.

Now, in (6) take C to be one point divisor Q and let $\phi = \phi(Q)$ be a principal part with a simple pole at Q. If ζ is an element of $H^1(X, \Omega(D))$ given by $\{c_{ij}\}$ of (7) as before, then ζ generates the kernel of β. For $\zeta' \in H^1(X, \Omega(D))$ is an element of the kernel of β if and only if ζ' is represented by $c'_{ij} = (f'_i/h_i) - (f'_j/h_j)$, where $(f'_i) > -Q$ in U_i, that is, $\{f'_i\}$ define a principal part ϕ' with a simple pole at Q or $\phi' = 0$, and then $\phi' = r\phi$ for some $r \in k$. Hence $\zeta = 0$ if and only if β is bijective, in other word, $H^0(X, \Omega(K - D - Q)) \cong H^0(X, \Omega(K - D))$, hence, if and only if Q is a fixed point of $|K - D|$. If Q is not a fixed point of $|K - D|$, then $\zeta \neq 0$ and $g \cup \zeta = 0$ for an element g of $H^0(X, \Omega(K - D - Q))$, that is, the divisor in $|K - D|$ defined by g contains Q, where $\cup$ is the cup product : $H^0(X, \Omega(K - D)) \times H^1(X, \Omega(D)) \longrightarrow H^1(X, \Omega(K))$.

Next suppose that $|K - D|$ has no fixed point.. Then, for any given point $Q \in X$, $\sigma(\phi(Q))$ is not a G_m-bundle defined by D by the above fact.

Thus we have a map $\sigma|X : X \longrightarrow P(H^1(X, \Omega(D))$ because if ϕ_1 and ϕ_2 are two principal parts with a simple pole at the same point of X, then $\sigma(\phi_1) = r\sigma(\phi_2)$ for some $r \in k^*$. Now, the points of $P(H^1(X, \Omega(D)))$ are in bijective correspondence with the hyperplane sections of the projective space of $|K - D|$. Let $N = \dim|K - D|$. Consider the projective model of X defined by a basis of $|K - D|$ in P^N, then P^N and $|K - D|$ are dual each other. Thus $P(H^1(X, \Omega(D)))$ can be identified with the above P^N and since $(\sigma|X)(Q)$ corresponds to hyper-planes of $|K - D|$ which contains Q, $(\sigma|X)(Q)$ is identified with $f(Q)$ by this identification, where $f : X \longrightarrow P^N$ is a regular map by a basis of $|K - D|$.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & P^N \\ \downarrow{\scriptstyle \sigma|X} & & \downarrow{\scriptstyle \text{dual}} \\ P(H^1(X, \Omega(D))) & \xrightarrow{\quad \text{dual} \quad} & |K - D| \end{array}$$

Our results are the following.

Theorem 2.2. Let $\mathcal{L}$ be the vector space of all principal part over X. Then there is a map $\sigma : \mathcal{L} \longrightarrow \gamma^{-1}(D)$ with the following properties ;

(i) σ is a surjective linear map.

(ii) $\eta \in \gamma^{-1}(D)$ has a cross-section with a principal part ϕ if and only if $\eta = \sigma(\phi)$.

(iii) For a point $Q \in X$, $\sigma(\phi(Q))$ is a G_m-bundle defined by D if and only if Q is a fixed point of $|K - D|$, where $\phi(Q)$ is a principal part with a simple pole at Q.

(iv) If $|K - D|$ has no fixed point, then $(\sigma|X)(X)$ is the projective model of X defined by $|K - D|$, where $(\sigma|X)(Q) = \sigma(\phi(Q))$ for a point $Q \in X$.

Corollary 2.3. If $\sigma(\phi(Q_1)), \sigma(\phi(0_2)), \ldots, \sigma(\phi(Q_r))$ are linearly dependent, that is, $\Sigma_{i=1}^{r}\lambda_i\sigma(\phi(Q_i)) = 0$ for some $\lambda_i \in k$, then $\sigma(\Sigma_{i=1}^{r}\lambda_i\phi(Q_i))$ is a G_m-bundle.

Proof. σ can be factored into the a linear map τ of $\mathcal{L}$ onto $H^1(X, \Omega(D))$ and the natural projection Φ of $H^1(X, \Omega(D))$ onto $\gamma^{-1}(D)$. Then $\Sigma_{i=1}^{r}\lambda_i\sigma(\phi(Q_i)) = 0$ if and only if $\tau(\Sigma_{i=1}^{r}\lambda_i\phi(Q_i)) = \Sigma_{i=1}^{r}\lambda_i\tau(\phi(Q_i)) = 0$. Thus our assertion is a direct consequence of Theorem 2.1. Q.E.D.

Let $Y(D)$ be the projective space consisting of A-bundles which are contained in $\gamma^{-1}(D)$ and not a G_m-bundle defined by D (this space can be identified with $P(H^1(X, \Omega(D)))$ by Theorem 2.1).

Corollary 2.4. $\eta \in Y(D)$ has a cross-section with poles among the points $Q_1, \ldots, Q_r$ if and only if there are principal parts $\phi_1, \ldots, \phi_s$ which have poles among the points $Q_1, \ldots, Q_r$, and η lies in the subspace of $Y(D)$ spanned by the points $\sigma(\phi_1), \ldots, \sigma(\phi_s)$.

Proof. This is immediate from (i), (ii) of Theorem 2.2.

Since $PGL(1)$ is the automorphism group of P^1, two A-bundles η and

η' are equivalent as P^1-bundles if and only if their bundle spaces can be identified. Let $\eta \in \gamma^{-1}(D)$ and $\eta' \in \gamma^{-1}(D')$ and let s and s' be the infinite cross-sections of η and η' respectively. If η is equivalent to η' as P^1-bundle, then s' is a section of bundle space of η, hence the divisor $s - s'$ is linearly equivalent to the sum of fibres (see Lemma 3.2). On the other hand, $\pi(s \cdot s) = -D$ and $\pi(s' \cdot s') = -D'$ by the equation (5) of § 1. Thus, if $\pi(s \cdot s') = n_1Q_1 + n_2Q_2 + \ldots + n_rQ_r$ ($n_1, \ldots, n_r$ are positive integers and $Q_i \neq Q_j$ if $i \neq j$.), then $0 = \pi((s - s') \cdot (s - s')) = \pi(s \cdot s) - 2\pi(s \cdot s') + \pi(s' \cdot s') = -D - 2(n_1Q_1 + \ldots + n_rQ_r) - D'$, that is, we have the formula

$$D + D' = -2(n_1Q_1 + n_2Q_2 + \ldots + n_rQ_r) \tag{8},$$

where $(n_1Q_1 + n_2Q_2 + \ldots + n_rQ_r)$ is the divisor class defined by the divisor $n_1Q_1 + n_2Q_2 + \ldots + n_rQ_r$. Suppose η is given by $\{a_{ij}, b_{ij}\}$. The infinite section s' of η' gives a section $\{f_i\}$ of η with poles of multiplicity n_i at Q_i. We may assume that the coordinate neighbourhoods $\{U_i\}$ satisfy the following two conditions,

(i) each Q_i is contained in only one of U_i, $i = 1, 2, \ldots, r$,

(ii) U_p does not contains any zero of f_p, for each $p = 1, 2, \ldots, r$.

Let us choose projective transformations as follows

$$w_q = (z_q - f_q)^{-1} \quad \text{in } U_q \text{ if } q \text{ is not any of } 1, 2, \ldots, r,$$

$$w_p = (f_p^{-1} - z_p^{-1})^{-1} \quad \text{in } U_p \quad \text{if } p = 1, 2, \ldots, r.$$

Then, since $z_i = f_i$ imples $w_i = \infty$, the coordinate transformations of can be taken as follows ;

$$w_q = (z_q - f_q)^{-1} = a_{qm}^{-1}(z_m - f_m)^{-1} = a_{qm}^{-1}w_m \quad \text{in } U_q \cap U_m \quad \text{if } q, m$$

$\neq 1. 2. \ldots. r$

$$w_p - f_p = f_p^2(z_p - f_p)^{-1} = a_{pm}^{-1}f_p^2(z_m - f_m)^{-1} = a_{pm}^{-1}f_p^2 w_m$$

in $U_m \cap U_p$ if $p = 1, 2, \ldots, r$ and $m \neq 1, 2, \ldots, r$,

$$w_p - f_p = f_p^2(z_p - f_p)^{-1} = a_{pt}^{-1}f_p^2(z_t - f_t)^{-1} = a_{pt}^{-1}f_p^2 f_t^{-2}(w_t - f_t)^{-1}$$

in $U_p \cap U_t$ if $p, t = 1, 2, \ldots, r$.

Thus $D' = \gamma(\eta')$ is given by $\{a'_{ij}\}$, where

$a'_{qm} = a_{qm}^{-1}$ if q, m are not any of $1, 2, \ldots, r$,

$a'_{pm} = a_{pm}^{-1}f_p^2$ if $p = 1, 2, .., r$ and if m is not any of $1, 2, .., r$,

$a'_{pt} = a_{pt}^{-1}f_p^2 f_t^{-2}$ if $p, t = 1, 2, \ldots, r$.

On the other hand, if $\psi(Q_i)$ is a fixed function on U_i with a simple pole at Q_i and no zero in U_i and $\phi(Q_i)$ is the principal part of $\psi(Q_i)$, then $\phi = \Sigma_{i=1}^{r} \Sigma_{k=1}^{n_i} \lambda_{ik}\phi(Q_i)^k$, where ϕ is the principal part of $\{f_i\}$,

λ_{ik} is a suitable constant and $\lambda_{in_i} \neq 0$. Making the transformations $w'_i = \psi(Q_i)^{2n_i f^{-2}} \cdot w_i$, we may replace the a'_{ij} given above by new ones independent of λ_{ik};

$$a'_{qm} = a_{qm} \quad \text{if} \quad q, m \neq 1, 2, \ldots, r,$$

$$a'_{pm} = a^{-1}_{pm}\psi(Q_p)^{2n_p} \quad \text{if} \quad p = 1, 2, \ldots, r \quad \text{and} \quad m \neq 1, 2, \ldots, r,$$

$$a'_{pt} = a^{-1}_{pt}\psi(Q_p)^{2n_p}\psi(Q_t)^{-2n_t} \quad \text{if} \quad p, t = 1, 2, \ldots, r.$$

And the coordinate transformations are

$$w'_q = a'_{qm} \cdot w'_m \,, \qquad w'_p = a'_{pm} \cdot w'_m + \psi(Q_p)^{2n_p f_p^{-1}},$$

$$w'_p - \psi(Q_p)^{2n_p f_p^{-1}} = a'_{pt}(w'_t - \psi(Q_t)^{2n_t f_t^{-1}}),$$

where q, m, p, t are as above. Thus there is a section of η' with principal part $\Sigma^r_{i=1} \Sigma^{n_i}_{k=1}\mu_{ik}\phi(Q_i)^k$, where $\Sigma^{n_i}_{k=1}\mu_{ik}\phi(Q_i)^k$ is the principal part of $\psi(Q_i)^{2n_i f_i^{-1}}$ in paticular $\mu_{in_i} = \lambda^{-1}_{in_i}$. Therefore, if $\phi(Q_1), \ldots, \phi(Q_r)$ are fixed and λ_{ik} vary, we shall get a correspondence $\eta \longrightarrow \eta'$, between A-bundles η and η' which are equivalent to each other as P^1-bundle, in which the subspace $Y(D)$ spanned by $\sigma(\phi(Q_1)), \ldots, \sigma(\phi(Q_1)^{n_1})$, $\ldots, \sigma(\phi(Q_r)), \ldots, \sigma(\phi(Q_r)^{n_r})$ corresponds to the subspace of $Y(D')$ spanned by $\sigma'(\phi(Q_1)), \ldots, \sigma'(\phi(Q_1)^{n_1}), \ldots, \sigma'(\phi(Q_r)), \ldots, \sigma'(\phi(Q_r)^{n_r})$. This correspondence is the following ;

$$\Sigma_{i=1}^{r} \Sigma_{k=1}^{n_i} \lambda_{ik} \sigma(\phi(Q_i)^k) \longrightarrow \Sigma_{i=1}^{r} \Sigma_{k=1}^{n_i} \mu_{ik} \sigma'(\phi(Q_i)^k) \qquad (9),$$

where notaitions are as above. In paticular, if $Q_1, \ldots, Q_r$ are simple poles of the section $\{f_i\}$, then we have $\mu_{i1} = \lambda_{i1}^{-1}$ and $n_i = 1$ for all i. The criterion that two A-bundles are equivalent to each other as P^1-bundle is that the equations (8) and (9) are satisfied.

§ 3. Comparison with the results of chapter I.

We shall consider first the results induced from the equations (5) and (8). If there is a section s of P^1-bundle P(E) such that $(s, s) < 0$, then $(s', s') > 0$ for any another section s' of P(E) because $(s, s) + (s', s') \geqq 0$ by the equations (5) and (8). This means that the section such that $(s, s) < 0$ is unique if P(E) has such a section. Thus we have Lemma 1.2 by Lemma 1.14. If a P^1-bundle has two sections which do not intersect each other, then this bundle is a G_m-bundle (Remark 1.20). Thus, if P(E) $(E \neq I^2)$ has a section s such that $(s, s) = 0$, then either s is the unique minimal section or P(E) is a G_m-bundle with two minimal sections, because the self-intersection number of an arbitrary section is not less than 0 by (8) and because if P(E) has two sections with self-intersection number 0, they do not intersect each other by (8). This proves our Corollary 1.6. Combining above results and Theorem 2.1, we have Theorem 1.11.

Chapter III. Elementary transformations.

In this chapter, we assume always that X is a complete non-singular curve of genus $g \geqq 1$ over k. For the rational case Natata [10] studied in full detail and our Theorem 3.8, 3.13 are, in fact, true in this case too. We shall study the classification problem of ruled surfaces with genus $g \geqq 1$ by the same method as in [9], [10]. We use the method of universal domain. π, π' denote always the natural projections of ruled surfaces onto X.

§ 1. General results.

Let S be a ruled surface. We shall denote the fibre passing through a point $P \in S$ by ℓ_P. Let dil_P and cont_ℓ be the dilatation at a point P of a surface and the contraction by an exceptional curve of the first kind ℓ respectively. Now, $\mathrm{dil}_P[\ell_P]$ is an exceptional curve of the first kind on $\mathrm{dil}_P\, S$, where $\mathrm{dil}_P[\ell_P]$ is the proper transform of ℓ_P by dil_P. Hence we can define $\mathrm{cont}_{\mathrm{dil}_P[\ell_P]}$. We denote $\mathrm{cont}_{\mathrm{dil}_P[\ell_P]} \cdot \mathrm{dil}_P$ by elm_P and call it the elementary transformation at P. It is clear that $\mathrm{elm}_P\, S$ is also a ruled surface and the fibre over $\pi(P)$ is $\mathrm{elm}_P(P)$. Conversely, every ruled surface can be obtained from the direct product $P^1 \times X$ by some successive elementary transformations, but since the proof of this fact is easy and written in the book [13, Chap.V, § 1, Th.1] we omit it.

Let S_0 be the direct product $P^1 \times X$ of the projective line P^1 and a curve X. In the sequal of these lectures, $\mathrm{elm}_{P_1,\ldots,P_n} S$ is the ruled surface which is obtained from a ruled surface S by a succession of elementary transformations at points*) $P_1,\ldots, P_n$ in a suitable order**). We must note that elm_{P_1,P_2} cannot be defined if P_1 and P_2 lie***) on the same fibre of S. We shall begin with an elementary lemma.

Lemma 3.1. Let $P_1,\ldots, P_s$ be points lying on a section $P \times X$ of S_0 and $\dim | \Sigma_{i=1}^{s} P_i | = d$. Then we have

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*) In this and next chapters a point may be an infinitely near point.

**) $\mathrm{elm}_{P_1,\ldots,P_{r-1}} \mathrm{elm}_{P_r}$ is defined if and only if P_r is an ordinary point of $\mathrm{elm}_{P_1,\ldots,P_{r-1}} S$.

***) A point P is said to lie on a positive divisor D if, after suitable successive quadratic dilatations, P becoems an ordinary point lying on the proper transform of D. A positive divisor D goes through points $P_1,\ldots, P_n$ if, after suitable successive dila dilatations, $D - \Sigma P_i$ becomes a positive divisor. Thus, lying on and going through have different meaning from each other. For instance, a curve c may go through P, with an infinitely near at P not lying on c.

$$\dim \mid r(P \times X) + \Sigma_{i=1}^{s} \ell_{P_i} \mid = (r + 1)d + r$$

where $r \geqq 0$.

Proof. We prove this lemma by induction on r. First note that $\Sigma\, \ell_{R_i} \sim \Sigma\, \ell_{R_i'}$ if and only if $\Sigma\, \pi(\ell_{R_i}) \sim \Sigma\, \pi(\ell_{R_i'})$ on X. When $r = 0$, we know that $\dim \mid \Sigma_{i=1}^{s} \ell_{P_i} \mid = \dim (\mathrm{Tr}_{P \times X} \mid \Sigma_{i=1}^{s} \ell_{P_i} \mid) + \dim (\mid \Sigma_{i=1}^{s} \ell_{P_i} - P \times X) + 1$. By the above remark we obtain $\dim (\mathrm{Tr}_{P \times X} \mid \Sigma_{i=1}^{s} \ell_{P_i} \mid) = \dim \mid \Sigma_{i=1}^{s} P_i \mid = d$. Moreover, no member of the linear system $|\Sigma \ell_{P_i}|$ contains the section $P \times X$ because $(\Sigma\, \ell_{P_i}, \ell_{P_1}) = 0$ and $(P \times X, \ell_{P_1}) = 1$. Hence $\dim \mid \Sigma_{i=1}^{s} \ell_{P_i} \mid = d - 1 + 1 = d$. Next assume $r \geqq 1$, then $\dim \mid r(P \times X) + \Sigma_{i=1}^{s} \ell_{P_i} \mid = \dim (\mathrm{Tr}_{P \times X} \mid r(P \times X) + \Sigma_{i=1}^{s} \ell_{P_i} \mid) + \dim \mid (r - 1)(P \times X) + \Sigma_{i=1}^{s} \ell_{P_i} \mid + 1$. On the other hand, $\dim (\mathrm{Tr}_{P \times X} \mid r(P \times X) + \Sigma_{i=1}^{s} \ell_{P_i} \mid) = d$ by the similar reason as in the case of $r = 0$ and $\dim \mid (r - 1)(P \times X) + \Sigma_{i=1}^{s} \ell_{P_i} \mid = rd + r - 1$ by our induction hypothesis. Thus our proof is completed.

The following lemma is very useful.

Lemma 3.2. If D is a positive divisor on S_0, then D is linearly equivalent to $r(P \times X) + \Sigma_{i=1}^{s} \ell_i$ for some fibres $\ell_1, \ldots, \ell_s$, where $r = (\ell_1, D)$ and $s = (P \times X, D)$.

Proof. Since it suffices to prove the assertion for each component of D, we may assume that D is an irreducible curve. By Lemma 3.1 we have $\dim | r(P \times X) + \Sigma_{i=1}^{s} \ell_i | = (r + 1)(t - g) + r$ if $t > 2g - 2$. On the other hand, $(D, r(P \times X) + \Sigma_{i=1}^{t} \ell_i) = rs + tr = r(s + t)$. Thus if t is sufficiently large,

$$\dim | r(P \times X) + \Sigma_{i=1}^{t} \ell_i | > (D, r(P \times X) + \Sigma_{i=1}^{t} \ell_i) = r(s + t).$$

Now, we fix the number t satisfying the above inequality. Then there is a member D' of $|r(P \times X) + \Sigma_{i=1}^{t} \ell_i|$ such that it goes through $r(s + t) + 1$ points on D. Since $(D, D') = r(s + t)$, we have that $D' = D + D''$ for some $D'' > 0$. Moreover, $D'' = \Sigma_{i=1}^{u} \ell'_i$ because $(D'', \ell_1) = (D' - D, \ell_1) = r - r = 0$. Hence we have that $r(P \times X) + \Sigma_{i=1}^{t} \ell_i - \Sigma_{j=1}^{u} \ell'_j \sim D$. If one takes the traces of them on a section $P' \times X$ ($P' \in P^1$), it is easy to see that $\Sigma_{i=1}^{t} \ell_i - \Sigma_{j=1}^{u} \ell'_j \sim \Sigma_{k=1}^{s} \ell''_k$ for some fibres $\ell''_1, \ldots, \ell''_s$. Q.E.D.

Remark 3.3. We get the following formula for the arithmetic genus by the genus formula.

$$p_a(r(P \times X) + \Sigma_{i=1}^{s} \ell_i) = rg + (r - 1)(s - 1).$$

Remark 3.4. D is a non-singular irreducible curve of genus g on S_0 if and only if D is an irreducible member of $|P \times X + \Sigma_{i=1}^{s} \ell_i|$ for suitable fibres $\ell_1, \ldots, \ell_s$.

Proof. If D is a non-singular irreducible curve of genus g, then

$D \sim r(P \times X) + \Sigma_{i=1}^{s} \ell_i$ by lemma 3.2. If $r > 1$, Remark 3.3 implies that $g + s - 1 = 0$ and therefore $g = 1$ and $s = 0$, that is, $D \sim r(P \times X)$ and D is irreducible. On the other hand, we have that $|r(P \times X)| = \{P_1 \times X + \ldots + P_r \times X \mid P_1, \ldots, P_r \in P^1\}$. This is a contradiction. Conversely, let D be an irreducible element of $|P \times X + \Sigma_{i=1}^{s} \ell_i|$. If D has singular points, then $(D, \ell) > 1$ and hence $D \not\sim P \times X + \Sigma\, \ell_i$. Therefore, D is a non-singular curve. Moreover, the genus of D is equal to g by Remark 3.3. Q.E.D.

Lemma 3.5. Let Q_i, R_j, S_1, T_m be points of X such that

(i) $\Sigma_{i=1}^{q} Q_i + \Sigma_{j=1}^{r} R_j$ is linearly equivalent to $\Sigma_{\ell=1}^{s} S_\ell + \Sigma_{m=1}^{q+r-s} T_m$,

(ii) $s \geqq g$ and $r + s \geqq 2g$,

(iii) $R_1, \ldots, R_r$ are independent generic points of X over k and all of Q_i and T_m are k-rational points. Then $\Sigma_{\ell=1}^{s} S_\ell$ is non-special in the sense of Riemann-Roch theorem.

Proof. If $r > g$, then considering $k(R_1, \ldots, R_{r-g})$ instead of k and $q + r - g$ instead of q, we may assume that $r = g$. Thus, in general, we may assume $r \leqq g$. We may assume that $\Sigma_\ell S_\ell$ is a generic member of $|\Sigma_\ell S_\ell|$ over $k(R_1, \ldots, R_r)$. Since $r \leqq g$, and $\Sigma\, R_j$ is non-special, we have $\dim |\Sigma\, R_j| = 0$. This, that the Q_i are k-rational points and that $|\Sigma\, Q_i + \Sigma\, R_j| = |\Sigma_\ell S_\ell + \Sigma\, T_m|$ is defined over $k(S_1, \ldots, S_s)$ show that $R_1, \ldots, R_r$ are algebraic over $k(S_1, \ldots, S_s)$. Therefore,

$\text{trans.deg}_k\, k(S_1,\dots, S_s) = \text{trans.deg}_k\, k(R_1,\dots, R_r, S_1,\dots, S_s) =$

$\text{trans.deg}_k\, k(R_1,\dots, R_r) + \text{trans.deg}_{k(R_1,\dots, R_r)}\, k(R_1,\dots, R_r, S_1,\dots, S_s)$

$\geqq r + \dim |\, \Sigma S_\ell \,| \geqq r + s - g \geqq g$. Thus, among $S_1,\dots, S_s$, there are at least g independent generic points of X over k, whence ΣS_ℓ is non-special. Q.E.D.

Now, we prove a key lemma which has a geometric meaning of Lemma 1.5. Let s_1 and s_2 be mutually distinct sections of S_0. We mean by $s_1 \cap s_2$ the set of common points, including infinitely near points, of s_1 and s_2. Assume that $D_1 = s_1 + \Sigma_i\, m_i \pi^{-1}(Q_i)$ is linearly equivalent to $D_2 = s_2 + \Sigma_j\, n_j \pi^{-1}(R_j)$, where (i) m_i and n_j are natural numbers, (ii) Q_i and R_j are points of X such that $Q_i \neq R_j$ for any i, j. Set $Q_i^* = (\pi^{-1}(Q_i)) \cdot s_2$ and $R_j^* = (\pi^{-1}(R_j)) \cdot s_1$. We consider sets of points as follows : For each Q_i^* or R_j^* (1) if $Q_i^* \in s_1 \cap s_2$ (or, if $R_j^* \in s_1 \cap s_2$, respectively), then let Q_i^{**} (or, R_j^{**}, resp.) be the infinitely near point of Q_i^* (or, R_j^*, resp.) of highest order among points in $s_1 \cap s_2$, and let M_i (or, N_j, resp.) be the set of infinitely near points of orders $1, 2, \dots, m_i$ (or, $1, 2,\dots, n_j$, resp.) of Q_i^{**} (or, R_j^{**}, resp.) such that they lie on s_2 (or, s_1, resp.) ; (2) if $Q_i^* \notin s_1 \cap s_2$ (or, if $R_j^* \notin s_1 \cap s_2$, respectively), then let M_i (or, N_j, resp.) be the set of infinitely near points of orders $0, 1,\dots, m_i - 1$ (or, $0, 1,\dots, n_j - 1$, resp.) of Q_i^* (or, R_j^*, resp.) such that they lie on s_2 (or, s_1, resp.). Now, rename the points of

$(s_1 \cap s_2) \cup (\bigcup M_i) \cup (\bigcup N_j)$, say $P_1, \ldots, P_u$, then we have

Lemma 3.6. In the above situation we have that $\mathrm{elm}_{P_1,\ldots,P_u} S_0$ is isomorphic to S_0 as P^1-bundle.

Proof. Consider the linear system $|D_1| - \Sigma_{i=1}^{u} P_i$ *). Since D_1 and D_2 are contained in $|D_1| - \Sigma P_i$, we obtain that $\dim (|D_1| - \Sigma P_i) \geqq 1$. We can see easily that all proper transforms of member of $|D_1| - \Sigma P_i$ do not intersect each other and they are sections of $\mathrm{elm}_{P_1,\ldots,P_u} S_0$, which completes our proof. **)

Corollary 3.7. Let $\Sigma_{i=1}^{r} m_i P_i$ and $\Sigma_{i=1}^{s} m_i' P_i'$ be divisors on $P \times X$ $P' \times X$ $(P, P' \in P^1, P \neq P')$ respectively. If $\Sigma\, m_i \pi(P_i) \sim \Sigma\, m_i \pi(P_i')$ on X and if $\pi(P_i) \neq \pi(P_j')$ for any i, j, then $\mathrm{elm}_{P_{11},\ldots,P_{rm_r}, P_1',\ldots,P'_{sm_s'}} S_0$ is isomorphic to S_0 as P^1-bundle, here $P_{i,\ell}$ (or, $P'_{j,t}$) is the infinitely near point of order $\ell - 1$ (or, $t - 1$, resp.) of P_i (or, P_j', resp.) which lies on $P \times X$ (or, $P' \times X$, resp.).

Proof. Put $D_1 = P \times X + \Sigma_{i=1}^{r} \ell_{P_i'}$ and $D_2 = P' \times X + \Sigma_{i=1}^{r} \ell_{P_i}$, then Lemma 3.6 is applicable.

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*) Let L be a linear system and P_i be points. Then $L - \Sigma P_i$ denotes the linear system which consists of members of L going through all P_i.

**) If a P^1-bundle has three sections which don't intersect each other, then it is isomorphic to the trivial bundle.

The following theorem which was proved in [12] offers a tool to classify ruled surfaces and to determine the structures of them.

Theorem 3.8. If S is a ruled surface, then S is isomorphic to $\mathrm{elm}_{P_1,\dots,P_n} S_0$ as P^1-bundle with suitable points $P_1,\dots,P_n$ on S_0 such that either (1) all the $P_1,\dots,P_n$ lie on a section $P \times X (P \in P^1)$ or (2) n is not greater than $2g + 1$.

Proof. Let $P_1,\dots,P_n$ be points such that S is isomorphic to $\mathrm{elm}_{P_1,\dots,P_n} S_0$. Assume that n is the smallest number of such points. We assume the contrary to the theorem for these points $P_1,\dots,P_n$ and shall derive a contradiction.

First step. Our assumption that n is the smallest implies

(i) If $R_1,\dots,R_t \in X$, $L_t = |P \times X + \Sigma_{i=1}^{t} \pi^{-1}(R_i)|$, and if $D_1, D_2 \in L_t - \Sigma_{i=1}^{t+1} P_{\alpha_i}$ $(\alpha_1 < \dots < \alpha_{t+1} \leq n)$, then D_1 and D_2 contains a common section of S_0.

Proof. If D_1 and D_2 contain a common fibre, then we can reduce to the case where t is one less since P_i and P_j $(i \neq j)$ do not lie on the same fibre. Now we assume that D_1 and D_2 contain no common component. Then, by Lemma 3.6, there are points $Q_1,\dots,Q_{t-1}$ on S_0 such that $\mathrm{elm}_{P_{\alpha_1},\dots,P_{\alpha_{t+1}},Q_1,\dots,Q_{t-1}} S_0$ is isomorphic to S_0 since

$(D_1, D_2) = 2t$. Consider the inverse*) of $\mathrm{elm}_{Q_1,\ldots,Q_{t-1}}$. Then we have that $\mathrm{elm}_{P_{\alpha_1},\ldots,P_{\alpha_{t+1}}} S_0$ is isomorphic to $\mathrm{elm}_{Q_1^*,\ldots,Q_{t-1}^*} S_0$ and this contradicts to the smallestness of n.

Second step. For $R_1,\ldots, R_{n-1} \in X$, we consider $L_{n-1} = |P \times X + \Sigma_{i=1}^{n-1}\pi^{-1}(R_i)|$. Since $n \geqq 2g + 2$ by our assumption $\dim |\Sigma R_i| = n - 1 - g$, whence we have $\dim L_{n-1} = 2n - 2g - 1$ by Lemma 3.1. Therefore,

$$\text{(ii)} \qquad \dim (L_{n-1} - \Sigma_{i=1}^{n} P_i) \geqq n - 2g - 1 \geqq 1,$$

for arbitrary $R_1,\ldots, R_{n-1} \in X$.

Let $P \in P^1$ be the point such that $P_1 \in P \times X$, and $Q_1,\ldots, Q_n \in X$ be points such that $\pi^{-1}(Q_i)$ goes through P_i for each i. We may assume $P_1,\ldots, P_s$ lie on $P \times X$ and none of $P_{s+1},\ldots, P_n$ does. Then we have $s \geqq n - 1$ by our assumption on the P_i.

Third step. Let $R_1,\ldots, R_{s-1}$ be independent generic points of X and $R_{s+i} = Q_{s+i+1}$ for $i \geqq 0$. Then $L_{n-1} - \Sigma_{i=1}^{n} P_i$ contains a member $D_1 = P \times X + \Sigma\, \pi^{-1}(R_i)$. By (i) above, every member of $L_{n-1} - \Sigma_{i=1}^{n} P_i$

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*) The inverse of elm_P is obtained as follows : If P is an ordinary point on a ruled surface S and $\mathrm{elm}_P\, S = S'$, then $Q = \mathrm{elm}_P\, [\ell_P]$ is an ordinary point on S' and $\mathrm{elm}_Q \cdot \mathrm{elm}_P\, S = \mathrm{elm}_Q\, S' = S$ and hence the inverse of elm_P is elm_Q.

contains $P \times X$, whence $L_{n-1} - \Sigma P_i = \{P \times X + \Sigma_{i=s+1}^{n} \pi^{-1}(Q_i) + \Sigma_{i=1}^{s-1} \pi^{-1}(S_i) \mid \Sigma_{i=1}^{s-1} S_i \in |\Sigma_{i=1}^{s-1} R_i|\}$. Since $\dim (L_{n-1} - \Sigma P_i) \geqq 1$ by (ii), we have $\dim |\Sigma_{i=1}^{s-1} R_i| \geqq 1$. On the other hand, $R_1, \ldots, R_{s-1}$ are independent generic points, so we have

(iii) $s \geqq g + 2$.

Fourth step.*) We may assume that the fixed points of $|\Sigma_{i=1}^{s} Q_i|$ are $Q_{u+1}, \ldots, Q_s$. Then we shall prove that

(iv) Every P_{s+i} $(i = 1, \ldots, n - s)$ is an infinitely near point of some P_j with $u + 1 \leqq j \leqq s$.

Proof. Assume for instance P_{s+1} is not an infinitely near point of any P_j $(u + 1 \leqq j \leqq s)$.

Case I. Let P_{s+1} be an ordinary point of S_0 lying on the section $P' \times X$ $(P' \neq P)$. Since $\dim |\Sigma_{i=1}^{u} Q_i| = \dim |\Sigma_{i=1}^{s} Q_i| \geqq s - g \geqq 2$ by (iii), there exists $Q_{s+1} + \Sigma_{j=2}^{u} Q_j^* \in |\Sigma_{i=1}^{u} Q_i|$. Now, consider $L_u = |P \times X + \Sigma_{i=1}^{u} \pi^{-1}(Q_i)|$, then $L_u - \Sigma_{i=1}^{u} P_i - P_{s+1}$ contains $P \times X + \pi^{-1}(Q_{s+1}) + \Sigma_{j=2}^{u} \pi^{-1}(Q_j^*)$ and $P' \times X + \Sigma_{i=1}^{u} \pi^{-1}(Q_i)$. This is a contradiction by (i).

Case II. Let P_{s+1} be an infinitely near points of some P_i with

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*) This step is not necessary. But we take this step because this is a good example of the method of elementary transformations.

$i \leqq u$. Take a general member $\Sigma_{i=1}^{u} Q_i^*$ of $|\Sigma_{i=1}^{u} Q_i|$ and let $P_i' = (P' \times X) \cdot (\pi^{-1}(Q_i^*))$ for some $P' \in P^1$ $(P \neq P')$. Then $elm_{P_1,\ldots, P_u, P_1',\ldots, P_u'} S_0$ is biregular to S_0 by Corollary 3.7. Now, consider the inverse of $elm_{P_1',\ldots, P_u'}$. It is of the form $elm_{P_1^*,\ldots, P_u^*}$ for the points P_i^* which lie on the proper transform of $P \times X$. Since P_{s+1} does not lie on the proper transform of $P \times X$, we can reduce to the case I on $elm_{P_1,\ldots, P_u, P_1',\ldots, P_u'} S_0$ with respect to $P_1^*,\ldots, P_u^*, P_{u+1},\ldots, P_n$. Thus we complete the proof of (iv).

The last step. We may assume that $P_1,\ldots, P_n$ are k-rational points. We consider a complete linear system

$$L = |\, P \times X + \Sigma_{i=1}^{s} \pi^{-1}(Q_i) + \Sigma_{i=1}^{n-s-1} \pi^{-1}(R_i) \,| ,$$

where $R_1,\ldots, R_{n-s-1}$ are independent generic points of X. By (iii) above, we have $\dim |\, \Sigma_{i=1}^{s} Q_i + \Sigma_{i=1}^{n-s-1} R_i \,| = n - 1 - g > n - s$, whence $\Sigma_{i=1}^{s} Q_i + \Sigma_{i=1}^{n-s-1} R_i \sim \Sigma_{i=s+1}^{n} Q_i + \Sigma_{i=1}^{s-1} S_i$ for some $S_i \in X$. Thus $L - \Sigma_{i=1}^{n} P_i$ contains $P \times X + \Sigma_{i=s+1}^{n} \pi^{-1}(Q_i) + \Sigma_{i=1}^{s-1} \pi^{-1}(S_i)$ by (ii). Therefore, every member of $L - \Sigma_{i=1}^{n} P_i$ contains $P \times X$ by (i).

On the other hand, Lemma 3.5 is applicable to

$$\Sigma_{i=1}^{s} Q_i + \Sigma_{i=1}^{n-s-1} R_i \sim \Sigma_{i=s+1}^{n} Q_i + \Sigma_{i=1}^{s-1} S_i$$

and so we have that ΣS_i is non-special. If $Q_{s+\alpha}$ $(\alpha > 0)$ is a fixed point of $|\Sigma_{i=1}^{s} Q_i + \Sigma_{i=1}^{n-s-1} R_i| - \Sigma_{i=s+1}^{s+\alpha-1} Q_i$, then $Q_{s+\alpha}$ is a fixed point of $| Q_{s+\alpha} + \Sigma S_i|$ and this contradicts to the fact ΣS_i is non-special. Therefore, we have

(v) $\dim | \Sigma_{i=1}^{s} Q_i + \Sigma_{j=1}^{n-s-1} R_j | - \dim (|\Sigma_{i=1}^{s} Q_i + \Sigma R_j| - \Sigma_{i=s+1}^{s+\alpha} Q_i)$

$= \alpha$ for every $\alpha = 1, 2, \ldots, n - s$.

Set $L_\alpha^* = L - \Sigma_{i=1}^{\alpha} P_i$ for $\alpha = 0, 1, \ldots, n$. If $\alpha \leqq s$, then P_α is a fixed point of $L_{\alpha-1}^*$ if and only if Q_α is a fixed point of $|\Sigma_{i=\alpha}^{s} Q_i + \Sigma_{j=1}^{n-s-1} R_j|$. Hence we have

(vi) $\dim L - \dim L_s^* = \dim | \Sigma_{i=1}^{s} Q_i + \Sigma_{j=1}^{n-s-1} R_j | - \dim | \Sigma R_j|$,

On the other hand, we know

(vii) $\dim L_\alpha^* = \dim (\mathrm{Tr}_{P \times X} L_\alpha^*) + \dim (L_\alpha^* - P \times X) + 1.$

Since $L_0^* = L$ and $L_s^* - P \times X = L - P \times X$, (vi) and (vii) for $\alpha =$ 0, s imply that $\dim (\mathrm{Tr}_{P \times X} L_s^*) = \dim | \Sigma_{j=1}^{n-s-1} R_j|$

Now, we consider the case where $\alpha > s$. Since every member of $L_\alpha^* -$ $P \times X$ contains $\Sigma_{i=s+1}^{\alpha} \pi^{-1}(Q_i)$, $\dim (L_\alpha^* - P \times X) = \dim (|\Sigma_{j=1}^{n-s-1} R_j + \Sigma_{i=1}^{s} Q_i$ $- \Sigma_{i=s+1}^{\alpha} Q_i)$. Thus, by (v) we have

(viii) $\dim (L^*_s - P \times X) - \dim (L^*_\alpha - P \times X) = \alpha - s$ (for $\alpha > s$).

Since $\dim L^*_s - \dim L^*_\alpha \leqq \alpha - s$, (vi) and (vii) for $\alpha = s$ and $\alpha > s$ imply that $\dim (\mathrm{Tr}_{P \times X} L^*_\alpha) = \dim (\mathrm{Tr}_{P \times X} L^*_s) \geqq 0$. In paticular, for $\alpha = n$, we know that

$$\dim L^*_n = \dim (\mathrm{Tr}_{P \times X} L^*_n) + \dim (L^*_n - P \times X) + 1 > \dim (L^*_n - P \times X).$$

This is a contradiction to the fact that every member of L^*_n contains $P \times X$. Thus, the proof of Theorem 3.8 is completed.

Recently, Nagata proved in [11] the following result which answers affirmatively the Atiyah's conjecture : $N(P(E))$ is at most g.

Lemma 3.9. Let $R_1, \ldots, R_{g+2d+1}$ ($d \geqq 0$) be arbitrary points on S_0. Then there is a divisor $D > 0$ on S_0 which is linearly equivalent to $P \times X + \Sigma_{i=1}^{g+d} \ell_i$ for some fibres $\ell_1, \ldots, \ell_{g+d}$ on S_0 and which goes through $R_1, \ldots, R_{g+2d+1}$.

To prove this lemma we need two sublemmas.

Lemma 3.10. If $\Sigma_{i=1}^{g+d} P_i$ is a generic member over k of a non-special complete linear system on $P \times X$ and D^* is a generic member of $|P \times X + \Sigma_{i=1}^{g+d} \ell_{P_i}|$ over $k(P_1, \ldots, P_{g+d})$, then $\dim_k k(D^*) = g + 2d + 1$.

Proof. Since $\Sigma_{i=1}^{g+d} P_i$ is a non-special divisor on $P \times X$, $\dim | \Sigma P_i | = d$ and so $\dim | P \times X + \Sigma_{i=1}^{g+d} \ell_{P_i} | = 2d + 1$ by Lemma 3.1.

Hence, $\dim_k k(P_1,\ldots, P_{g+d}, D^*) = \dim_k k(P_1,\ldots, P_{g+d}) +$ $\dim_{k(P_1,\ldots, P_{g+d})} k(P_1,\ldots, P_{g+d}, D^*) = g + d + 2d + 1 = g + 3d + 1$. Now, consider the locus T of $(D^*, P_1,\ldots, P_{g+d})$ over k, then $\dim_k T =$ $g + 3d + 1$. $(D^*, P'_1,\ldots, P'_{g+d})$ is in T if and only if $\Sigma P_i \sim \Sigma P'_i$. For, if two divisors D_1, D_2 are linearly equivalent each other, and if (D'_1, D'_2) is a specialization of (D_1, D_2), then $D'_1 \sim D'_2$. In paticular, since $(D^*, P \times X + \Sigma_{i=1}^{g+d} \ell_{P'_i})$ is a specialization of $(D^*, P \times X + \Sigma_{i=1}^{g+d} \ell_{P_i})$ over the specialization $(D^*, P_1,\ldots, P_{g+d}) \longrightarrow (D^*, P'_1,\ldots, P'_{g+d})$ and since $D^* \sim P \times X + \Sigma_{i=1}^{g+d} \ell_{P_i}$, we have that $\Sigma P_i \sim \Sigma P'_i$ if $(D^*, P'_1,\ldots, P'_{g+d})$ is a specialization of $(D^*, P_1,\ldots, P_{g+d})$. Conversely, since ΣP_i is a generic member of $|\Sigma_{i=1}^{g+d} P_i|$ over $k(D^*)$, $(D^*, P'_1,\ldots, P'_{g+d})$ is a specialization of $(D^*, P_1,\ldots, P_{g+d})$ if $\Sigma P_i \sim \Sigma P'_i$. Thus we get $\dim k(D)$ $= g + 2d + 1$ since $\dim |\Sigma P_i| = d$. Q.E.D.

Lemma 3.11. If $R_1,\ldots, R_n$ are points on a surface and $\dim_k k(R_1,\ldots, R_n) \geqq 2n - r$, then there are at least $(n - r)$ independent generic points of the surface in $\{R_1,\ldots, R_n\}$.

Proof. It is clear when $n = 1$. Now suppose $n \geqq 2$. If $\dim_{k(R_1,\ldots, R_{n-1})} k(R_1,\ldots, R_n) = 2$, there are $(n - 1 - r)$ independent generic points in $\{R_1,\ldots, R_{n-1}\}$ by our induction hypothesis, hence R_n and they are $(n - r)$ independent generic points. If $\dim_{k(R_1,\ldots, R_{n-1})} k(R_1,\ldots, R_n) = 1$, then $\dim_k k(R_1,\ldots, R_{n-1}) \geqq 2(n - 1) - (r - 1)$. Hence there are $(n - 1) - (r - 1) = n - r$ independent generic points in $\{R_1,\ldots, R_{n-1}\}$ by our induction hypothesis.

Now, go back to the proof of Lemma 3.9.

Proof of Lemma 3.9. Let D^* be the same as in Lemma 3.10. Let $R_1^*,\ldots, R^*_{2g+2d+1}$ be independent generic points of D^* (note that D^* is irreducible) over $k(D^*)$ and consider the locus T of $(D^*, R_1^*,\ldots, R^*_{2g+2d+1})$ over k. Then $\dim_k T = 3g + 4d + 2$ by Lemma 3.10. Let P_r be the projection morphism of T into the direct product of $(2g + 2d + 1)$ copies of S_0. If $(R_1',\ldots, R'_{2g+2d+1})$ is in $Pr(T)$, then $Pr^{-1}(R_1',\ldots, R'_{2g+2d+1})$ consists only of one element because if $(D_1', R_1',\ldots, R'_{2g+2d+1})$ and $(D_2', R_1', \ldots,R'_{2g+2d+1})$ are in $Pr^{-1}(R_1',\ldots,R'_{2g+2d+1})$, then $2g + 2d = (D^*, D^*) = (D_1', D_2') \geqq 2g + 2d + 1$ and it is a contradiction. Thus we get

$$\dim_k k(R_1^*,\ldots, R^*_{2g+2d+1}) = \dim_k Pr(T) = \dim_k T = 3g + 4d + 2.$$

Now, apply Lemma 3.11 to this case $(n = 2g + 2d + 1,\ r = g)$, then there are $g + 2d + 1$ independent generic points in $\{R_1^*,\ldots, R^*_{2g+2d+1}\}$, say $R_1^*,\ldots, R^*_{g+2d+1}$. Let T' be the locus of $(D^*, R_1^*,\ldots, R^*_{g+2d+1})$ and F be the direct product of $(g + 2d + 1)$ copies of S_0. Then, if Pr' is the projection of T' into F, Pr' is surjective. Therefore, there is a specialization D of D^* which goes through $R_1, \ldots, R_{g+2d+1}$ and by Lemma 3.2 this is a divisor for which we look. Q.E.D.

Before proving that g is the lowest upper bound of $N(P(E))$, we must show the following lemma.

Lemma 3.12. Let D^* be the same as in Lemma 3.10. If $\Sigma_{i=1}^{g+d} R_i$ is non-special divisor on $P \times X$, then each member of $| P \times X + \Sigma \ell_{R_i} |$ is a specialization of D^* over k.

Proof. Let C be the Chow point of $| P \times X + \Sigma_{i=1}^{g+d} \ell_{P_i} |$, which is regarded as a subvariety in the locus of D^* over k. Consider the specialization $(C, D^*, P_1, \ldots, P_{g+d}) \longrightarrow (C', D', R_1, \ldots, R_{g+d})$ over k, then D' is a member of $| P \times X + \Sigma_{i=1}^{g+d} \ell_{R_i} |$. Since D^* is contained in $\mathrm{Supp}(C)$, D' is an element of $\mathrm{Supp}(C')$ and so $\mathrm{Supp}(C')$ is contained in $| P \times X + \Sigma \ell_{R_i} |$. Because dimension of the support remains unchanged under the specialization process and $\dim | P \times X + \Sigma_{i=1}^{g+d} \ell_{R_i} | =$ $\dim | P \times X + \Sigma_{i=1}^{g+d} \ell_{P_i} |$ by our hypothesis, $\mathrm{Supp}(C') = | P \times X + \Sigma \ell_{R_i} |$ and each member of $\mathrm{Supp}(C')$ is a specialization of D^*. Q.E.D.

Theorem 3.13. The lowest upper bound of $N(P(E))$ is g.

Proof. First note that every section of $P(E)$ is the proper transform of a member of $| P \times X + \Sigma \ell_i |$ by $\mathrm{elm}_{R_1, \ldots, R_s}$ if $P(E) = \mathrm{elm}_{R_1, \ldots, R_s} S_0$, where $\ell_1, \ldots, \ell_r$ are suitable fibres of S_0. If $s \leqq g$, all $P \times X$ $(P \in P^1)$ of S_0 are transformed into sections whose self-intersection numbers are at most g. If $s = g + 2d$ or $g + 2d + 1$ $(d \geqq 0)$, then take a divisor D as in Lemma 3.9 for $R_1, \ldots, R_s$. If $D = D' + \Sigma_{i=1}^{t} \ell_i$ and D' is irreducible, then at least $(s - t)$ points of $\{R_1, \ldots, R_s\}$ lie on D' and $(D', D') = 2g + 2d - 2t$ since any two points of $\{R_1, \ldots, R_s\}$ don't lie on the same fibre. Thus self-intersection number of the proper transform of D' is at most $2g + 2d - s$, which is g or $g - 1$ according to $s = g + 2d$ or $g + 2d + 1$. Therefore, we obtain from Lemma 1.15 that $N(P(E)) \leqq g$. Next, if $d \geqq g - 1$, then each divisor of such a form $\Sigma_{i=1}^{g+d} R_i$ on $P \times X$ is non-special and so by Lemma 3.12 every divisor on S_0 which is linearly equivalent to some $P \times X + \Sigma_{i=1}^{g+d} \ell_i$ is a specialization of D^* in Lemma 3.10. Let T be the locus of $(D^*, R_1^*, \ldots, R_{g+2d+2}^*)$, where $R_1^*, \ldots, R_{g+2d+2}^*$ are independent generic points of D^* over $k(D^*)$. Then, $\dim T = 2g + 4d + 2$ by Lemma 3.10, whence the projection Pr of T into the direct product F of $(g + 2d + 2)$ copies of S_0 is never surjective. Thus $F - Pr(T)$ is non-empty Zariski open. It is easy now to see that $N(\mathrm{elm}_{R_1', \ldots, R_{g+2d+2}'} S_0)$ $= g$ for $(R_1', \ldots, R_{g+2d+2}') \in F - Pr(T)$. Q.E.D.

Let D be a divisor class on X with degree $D = n > 0$. By virtue of Theorem 2.2 (iv) we obtain a morphism $\sigma_D : X \longrightarrow P(H^1(X, L(-D)))$. On the other hand, there are rational functions $f_0(u, x), \ldots, f_{g+n-2}(u, x)$

on $J_n \times X$ such that $\{f_0(D', x), \ldots, f_{g+n-2}(D', x)\}$ is a basis of $H^0(X, L(K + D'))$ for any point of a suitable Zariski open set U_D in J_n which contains D, where J_n is the Jacobian variety of degree n. Take the dual basis $v_0(D'), \ldots, v_{g+n-2}(D')$ of $f_0(D', x), \ldots, f_{g+n-2}(D', x)$ in $H^1(X, L(-D'))$. Then we have a bijective map

$$h_D : U_D \times P^{g+n-2} \longrightarrow \{(D', \xi) \mid D' \in U_D, \ \xi \in P(H^1(X, L(-D)))\} ,$$

$$h_D(D', \alpha_0, \ldots, \alpha_{g+n-2}) = (D', \alpha_0 v_0(D') + \ldots + \alpha_{g+n-2} v_{g+n-2}(D')).$$

Therefore, we have P^{g+n-2}-bundle space V_n over J_n and a morphism $\tilde{\sigma} : J_n \times X \longrightarrow V_n$ such that $p^{-1}(D) \cong P(H^1(X, L(-D)))$ and $\tilde{\sigma} \mid (D) \times X = \sigma_D$ for any $D \in J_n$, where p is the projection of V_n to J_n.

If $N(P(E)) = n > 0$, then a vector bundle of canonical type E is an extension

$$0 \longrightarrow I \longrightarrow E \longrightarrow L \longrightarrow 0 ,$$

where $\deg L = n$, $L \in \mathcal{S}(P(E))$. Hence $P(E)$ determines some points of V_n (the correspondence is the same as in Theorem 1.11). The set of points of V_n which correspond to $P(E)$ is bijective to that of minimal sections of $P(E)$. Conversely if $(D, \xi) \in V_n$ is given, we have an extension

$$0 \longrightarrow I \longrightarrow E(D, \xi) \longrightarrow L(D) \longrightarrow 0 .$$

Hence we get a P^1-bundle $P(E(D, \xi))$ but $N(P(E(D, \xi)))$ is not always equal to n.

Now how many minimal sections do exist on a ruled surface S such that $N(S) > 0$? In the first place, consider the case where $N(S) = g$. We may assume that there are ordinary points $P_1, \ldots, P_{g+2d+2}$ and $S = \mathrm{elm}_{P_1, \ldots, P_{g+2d+2}} S_0$ for sufficiently large integer $d \geqq g - 1$.*) $(P_1, \ldots, P_{g+2d+2})$ must be contained in the set $F - Pr(T)$ in the proof of Theorem 3.13. A minimal section S is the proper transform of a section s of S_0 such that $s + \Sigma_{i=1}^{u} \ell_i' \in (|P \times X + \Sigma_{i=1}^{g+d+1} \ell_i| - \Sigma_{i=1}^{g+2d+2} P_i)$. Lemma 3.12 implies that every positive divisor which is linearly equivalent to $P \times X + \Sigma_{i=1}^{g+d+1} \ell_i$ for some fibres $\ell_1, \ldots, \ell_{g+d+1}$ is a specialization of a divisor $\bar{D}$. Let $\bar{R}_1, \ldots, \bar{R}_{g+2d+2}$ be the independent generic points of $\bar{D}$ over $k(\bar{D})$ and T' be the locus of $(\bar{D}, \bar{R}_1, \ldots, \bar{R}_{g+2d+2})$ over k. Then $\dim_k T' = 2g + 4d + 5$. If F' is the direct product of $(g + 2d + 2)$ copies of S_0, then $\dim F' = 2g + 4d + 4$ and the natural projection Pr' of T' to F' is surjective (see the proof of Lemma 3.9). Thus we know that every minimal section is the proper transform of an element of $Pr'^{-1}(P_1, \ldots, P_{g+2d+2})$, and that $\dim Pr'^{-1}(P_1, \ldots, P_{g+2d+2}) \geqq 1$.

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*) There is an integer d such that every ruled surface S is obtained by elementary transformations at $g + 2d$ or $g + 2d + 1$ ordinary points on S_0 (use Corollary 3.7 and Theorem 3.8).

On the other hand, if $\dim Pr'^{-1}(P_1,\ldots, P_{g+2d+2}) \geqq 2$, we can find a divisor in $Pr'^{-1}(P_1,\ldots, P_{g+2d+2})$ going through mutually distinct infinitely near points Q_1, Q_2 of an ordinary point of $P_1,\ldots, P_{g+2d+2}$ (see [11] Th. A1) and this divisor is of such a form $D + \ell$ that D goes through all of $\{P_1,\ldots, P_{g+2d+2}\}$. Then the proper transform of the prime component of D such that it is a section of S_0 is a section s with $(s, s) < g$. This is a contradiction. Thus we have

Proposition 3.14. If $N(S) = g$, then the set of minimal sections of a ruled surface S is of dimension one.

Let $\bar{D}_t$ be such a divisor that is defined in Lemma 3.10 for t points $P_1,\ldots, P_t$ and let $T_{t,r}$ be the locus of $(\bar{D}_t, R_1,\ldots, R_r)$ over k, where $R_1,\ldots, R_r$ are independent generic points of $\bar{D}_t$ over $k(\bar{D}_t)$ and $2t - r > 0$. Consider two closed subsets $G_{t,r}$, $H_{t,r}$ of $T_{t,r}$:

$$G_{t,r} = \{(D, Q_1,\ldots, Q_r) \mid (D, Q_1,\ldots, Q_r) \in T_{t,r}, \pi(Q_i) = \pi(Q_j) \text{ for some } i \neq j\}$$

$H_{t,r}$ = the locus of $(\bar{D}_{t-1} + \ell, R'_1,\ldots, R'_r)$ over k, where ℓ is a generic fibre over $k(\bar{D}_{t-1}, R'_1,\ldots, R'_r)$ and where $R'_1,\ldots, R'_r$ are independent generic points of $\bar{D}_{t-1}$.

Now, the map $f_{t,r}$ of $Z_{t,r} = T_{t,r} - (G_{t,r} \cup H_{t,r})$ to V_{2t-r}

$$f_{t,r} : (D, Q_1, \ldots, Q_r) \longrightarrow (\pi(D\cdot D) - \sum_{i=1}^{r} (Q_i), \xi)$$

is a rational map, where ξ is the element of $p^{-1}(\pi(D\cdot D) - \sum_{i=1}^{r} \pi(Q_i))$ determined by $\mathrm{elm}_{Q_1,\ldots, Q_r} S_0$ which can be regarded as A-bundle in $\gamma^{-1}(\Sigma\ \pi(Q_i) - \pi(D\cdot D))$. If r is sufficiently large, $f_{t,r}$ is a surjective map. It is easy to see that $N(\mathrm{elm}_{Q_1,\ldots, Q_r} S_0) < 2t - r$ if and only if $(Q_1,\ldots, Q_r) \in Pr_{t-1,r} (T_{t-1,r})$, where $Pr_{t,r}$ is the projection of $T_{t,r}$ to $S_0 \times \ldots \times S_0$ (r-ple product of S_0). Thus the image of $Pr^{-1}_{t,r} (P_{r_{t-1,r}} T_{t-1,r}) \cap Z_{t,r}$ by $f_{t,r}$ is just the subset of V_{2t-r} which consists of elements corresponding to P'-bundle with $N(P(E)) < 2t - r$. Therefore, we have

Lemma 3.15. The subset of V_n which consists of elements corresponding to P^1-bundles with $N(P(E)) = n$ is a dense subset of V_n which contains a Zariski open subset.

If $(D, \xi) \in V_2$ corresponds to $P(E)$ with $N(P(E)) < 2$, then $P(E)$ has sections s, s' such that $(s, s) = 2$, and $(s', s') < 2$. By virtue of Proposition 1.18, $(s', s') = 0$ or -2. Moreover if $(s', s') = -2$, then $(s, s') = \frac{1}{2} \{(s, s) + (s', s')\} = 0$ and so $P(E)$ is the G_m-bundle defined by D (see Remark 1.20), whence $P(E)$ cannot correspond to a point of V_2. Thus $(s', s') = 0$, which implies that s' is a section with simple pole at $Q = \pi(s\cdot s')$ since $2\pi(s\cdot s') = \pi(s, s) + \pi(s'\cdot s')$. Therefore,

(D, ξ) is contained in the image of $\tilde{\sigma}$ by Theorem 2.2(ii). Conversely if (D, ξ) is contained in the image of $\tilde{\sigma}$, then $P(E)$ has a section s' with simple pole and so $N(P(E)) \leqq (s', s') = 0$. On the other hand, $\tilde{\sigma}$ is proper. Hence the subset of V_2 which consists of elements corresponding to P'-bundles with $N(P(E)) = 2$ is a Zariski open set of V_2.

Next let us consider the case where $N(P(E)) = 1$. The following lemma implies that every element of V_1 corresponds to a P^1-bundle with $N(P(E)) = 1$.

Lemma 3.16. let L be a line bundle of degree 1 over X with $g \geqq 1$. Then a non-trivial extension E of L by I is a canonical type.

Proof. If E has a subbundle L' of $\deg L' \geqq 1$, we have $\deg (L \otimes L'^{-1}) = 0$ since $H^0(X, \underline{L} \otimes \underline{L}'^{-1}) \neq 0$ by Lemma 1.3. Thus $L \otimes L'^{-1} \cong I$ and so $L \cong L'$. Then, by Lemma 1.4, $E \cong I \oplus L \cong I \oplus L$ and this contradicts to the fact that E is a non-trivial extension. Q.E.D.

If $P(E)$ with $N(P(E)) = 1$ has minimal sections s, s', then s determines one point (D, ξ) in V_1 and (D, ξ) is the image of (D, Q) by $\tilde{\sigma}$ where $Q = \pi(s, s')$ (see Theorem 2.2). Thus the number of minimal sections of $P(E)$ is equal to $(\# \tilde{\sigma}^{-1} (D, \xi)) + 1$. By virtue of Theorem 2.2 we have $\# (\tilde{\sigma}^{-1} (D, \xi)) \leqq \deg (K + D) = 2g - 1$. Consequently we have the following theorem.

Theorem 3.17. Let n be a positive integer not greater than g.

(i) The subset U_n of V_n which consists of elements corresponding to P^1-bundles with $N(P(E)) = n$ is a dense subset of V_n which contains a Zariski open subset of V_n.

(ii) If $n = 2$, then U_2 is a Zariski open set of V_2.

(iii) If $n = 1$, then $U_1 = V_1$ and there is a surjective map $f_1 : V_1 \longrightarrow \mathfrak{P}^1_x = \{P(E) \mid N(P(E)) = 1\}$ such that $\# f_1^{-1}(P(E)) = \# \Delta(P(E)) \leqq 2g$ for all $P(E)$. Moreover, there is a closed set F of V_1 such that $\dim (P^{-1}(D) \cap F) = 1$ for all $D \in J_1$ and that any element of $f_1(V_1 - F)$ has only one minimal section.

Corollary 3.18. There is a surjective map $f_g : U_g \longrightarrow \mathfrak{P}^g_x = \{P(E) \mid N(P(E)) = g\}$ and $\dim f^{-1}(P(E)) = 1$ for all $P(E) \in \mathfrak{P}^g_x$.

Proof. By virtue of above theorem and Proposition 3.14 the proof is obvious.

We can prove easily the following fact : If $S = \mathrm{elm}_{P_1, \ldots, P_r} S_0$, $0 < N(S) < g$ and if $P_1, \ldots, P_r$ are sufficiently general, then the number of minimal sections of S $(= \# \Delta(S))$ is finite.

Here we present a conjecture.

Conjecture. If $N(S) < g$ and $S \not\cong S_0$, then $\# \Delta(S)$ is finite.

This is true if $g \leqq 3$ (use Theorem 3.8).

§ 2. Special cases.

We give, here, a rough classification of ruled surfaces in the case where $g = 1$ or 2.

For a ruled surface $S = \mathrm{elm}_{P_1, \ldots, P_r} S_0$, a section of S is the proper transform by $\mathrm{elm}_{P_1, \ldots, P_r}$ of an irreducible member of $|P \times X + \Sigma \ell_i|$ for some fibres ℓ_i because a section of S_0 is an irreducible member of such a linear system by Lemma 3.2. Thus we have the following tables of a classification if we take the results of the preceding section into account. We call a section of such a form $P \times X$ a base of S_0.

Table I. The case where $g = 1$.

$N(P(E))$	$\#\Delta(P(E))$	Type
1	∞	(1, 1, 1)
0	∞	the direct product.
0	1	(1, 1). One point is an infinitely near point of another which does not lie on a base.
0	2	(1, 1). Both points are ordinary points.
-1	1	(1).
-2	1	(2).
.	.	...
-n	1	(n).

In this table $(n_1, n_2, \ldots, n_r)$ expresses a surface which is obtained by the elementary transformations from the direct product S_0 at the points $P^1_1, P^1_2, \ldots, P^1_{n_1}, \ldots, P^r_1, P^r_2, \ldots, P^r_{n_r}$ such that $P^k_1, \ldots, P^k_{n_k}$ lie on the same base unless $n_k = 1$ and P^k_1 is an infinitely near point which does not lie on a base.

Table II. The case where $g = 2$.

Notations are the same as in the Table I.

$N(P(E))$	$\# \mathcal{A}(P(E))$	Type
2	∞	(1, 1, 1, 1).
1	at most 4	(1, 1, 1), (2, 1, 1, 1) or (2, 2, 1).
0	∞	The direct product.
0	2	(1, 1). Both points are ordinary points.
0	1	(1, 1). One point is an infinitely near point of another and both don't lie on a base.
0	1	(2, 1, 1).
0	2	(2, 2).
-1	1	(1).
-1	1	(2, 1).
-2	1	(2).
-2	1	(3, 1). The latter point is an infinitely near point of the former and doesn't lie on a base.
-3	1	(3).
-4	1	(4).
.	.	...
-n	1	(n).

Chapter IV. Models and special cases.

In this chapter we shall study the classification of P^1-bundles and their models in special cases.

§ 1. Decomposable (G_m-) bundles.

It is clear that a decomposable bundle in Chapter I is the same as a G_m-bundle in Chapter II. On the other hand, the following proposition shows the relation between decomposable bundles and elementary transformations.

Proposition 4.1. $P(E)$ is decomposable (i.e. E is so) if and only if $P(E)$ is obtained by the following way : Let $L \in \mathcal{L}(P(E))$ amd D_0 (or, D_∞, respectively) be the zero (or, polar, resp.) divisor of a divisor defined by the line bundle L. Take D_0 (or, D_∞, respectively) on $(0) \times X$ (or, $(\infty) \times X$, resp.) and perform the elementary transformations at such points on S_0, then the transformed surface is $P(E)$.

To prove this proposition we need a lemma which will be also used later.

Lemma 4.2. Let P be a point on a ruled surface S and put $T = \mathrm{elm}_P$. If s and s' are sections of S such that P lies on s and not on s', then $\pi'(T[s] \cdot T[s]) = \pi(s \cdot s) - \pi(P)$ and $\pi'(T[s'] \cdot T[s']) = \pi(s' \cdot s') + \pi(P)$ as divisor classes on X, where $T[s]$ (or, $T[s']$, respectively) is the proper transform of s (or, s', resp.) and where

π (or, π', respectively) is the natural projection of S (or, $T(S)$, resp.) onto X.

Proof. Since s and s' are sections, there are fibres $\ell_1, \ldots, \ell_r, \ell_{r+1}, \ldots, \ell_t$ such that $s \sim s' + \Sigma_{i=1}^{r} \ell_i - \Sigma_{j=r+1}^{t} \ell_j$ (see Lemma 3.2). Let ℓ be the fibre $T(P)$ of $T(S)$ and $T[\ell_i]$ be the proper transform of ℓ_i or $T[P]$ according to $\ell_i \not\ni P$ or $\ell_i \ni P$. Then we have that $T[s] + \ell \sim T[s'] + \Sigma_{i=1}^{r} T[\ell_i] - \Sigma_{j=r+1}^{t} T[\ell_j]$ and $\pi'\{T[s] \cdot (T[s'] + \Sigma_{i=1}^{r} T[\ell_i] - \Sigma_{j=r+1}^{t} T[\ell_j])\} = \pi(s \cdot s)$. Thus $\pi(T[s] \cdot T[s]) = \pi'\{T[s] \cdot (T[s'] + \Sigma_{i=1}^{r} T[\ell_i] - \Sigma_{j=r+1}^{t} T[\ell_j] - \ell)\} = \pi(s \cdot s) - \pi'(T[s] \cdot \ell) = \pi(s \cdot s) - \pi(P)$ and similarly $\pi'(T[s'] \cdot T[s']) = \pi(s' \cdot s') + \pi(P)$.

Q.E.D.

Proof of Proposition 4.1. To prove the "only if" part let $D_0 = \Sigma_{i=1}^{r} m_i P_i$, $D_\infty = \Sigma_{j=1}^{t} n_j Q_j$ (m_i, n_j are positive integers) and $z_i = a_{ij} z_j$ be the coordinate transformations of $P(E)$. Then L is defined by matrices (a_{ij}^{-1}) and so if s_∞ (or, s_0, respectively) is the infinite (or, zero, resp.) section, then $\pi(s \cdot s) = D_0 - D_\infty$ and $\pi(s_0 \cdot s_0) = D_\infty - D_0$. Let M_i (or, N_j, respectively) be the set of infinitely near points of order $0, 1, \ldots, m_i - 1$ (or, $0, 1, \ldots, n_j - 1$, resp.) of $P_i^* = \pi^{-1}(P_i) \cdot s_\infty$ (or, $Q_j^* = \pi^{-1}(Q_j) \cdot s_0$, resp.) such that they lie on s_∞ (or, s_0, resp.). Then rename the points of $(\cup M_i) \cup (\cup N_j)$, say $R_1, \ldots, R_u$. Now, consider the elementary transformation $T = \mathrm{elm}_{R_1, \ldots, R_u}$. Then $\pi'(T[s_\infty] \cdot T[s_\infty]) = 0$

and $\pi'(T[s_0] \cdot T[s_0]) = 0$ by Lemma 4.2 and $T[s_\infty]$ doesn't intersect $T[s_0]$. Thus $T(P(E))$ is the direct product of P^1 and X by Theorem 1.11 and Remark 1.20. Moreover, T^{-1} is the transformation stated in our proposition. Conversely, the surface $P(E)$ in our proposition has two sections which don't intersect each other, that is to say, the proper transforms of the sections $(0) \times X$, $(\infty) \times X$. And we get by Lemma 4.2 that $\pi(s_\infty) = D_0 - D_\infty$. Thus $P(E)$ is decomposable and $L \in \mathfrak{A}(P(E))$. Q.E.D.

§ 2. Special cases.

Now, we shall consider some special cases.

(A). The case where $g = 0$.

Since $2g - 2 = -2$, every $P(E)$ is decomposable by virtue of Remark 1.13 and Proposition 1.9. Hence $\mathfrak{P}_X = \mathfrak{P}_X^-$ and $C_X = \{(\mathfrak{A}(P(E)), 0)\}$. Since X is rational, $\mathfrak{A}(P(E))$ is determined by $N(P(E))$, that is to say, C_X is regarded as the set of non-positive integers. By Proposition 4.1, if $P(E)$ corresponds to n, then $P(E)$ is Nagata's F_{-n} [10]. Thus we have

Theorem 4.3. Isomorphism classes of P^1-bundles over a rational curve correspond bijectively to non-positive integers. If $P(E)$ corresponds to n, then $P(E)$ is nothing but the Nagata's F_{-n}.

Since $N(P(E))$ is an invariant of biregular classes of ruled surfaces, the above classification is that of biregular classes. By Corollary 1.7

and Proposition 1.18, F_{-n} $(n < 0)$ has only one minimal section (the self-intersection number being n) and another section has the self-intersection number not less than $-n$. This is analogous to the Proposition 2 of [10] and it can be proved easily from the above remark.

(B). The case where $g = 1$.

We need some lemmas to classify P^1-bundles over X with $g = 1$.

Lemma 4.4. Let $\mathcal{E}(2, 1)$ be the set of isomorphism classes of indecomposable vector bundles of rank 2 with degree 1. Then every element of $\mathcal{E}(2, 1)$ is of canonical type. Therefore, we have $N(E) = 1$ for all $E \in \mathcal{E}(2, 1)$.

Proof. Let $E \in \mathcal{E}(2, 1)$. Then, by the Riemann-Roch theorem on X we have $\dim H^0(X, E) \geqq 1$. Hence there is a grobal section $\phi \in H^0(X, E)$. However, if a line bundle L of positive degree is a subbundle of E, then we have an exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow (\det E) \otimes L^{-1} \longrightarrow 0$$

and this means that E is decomposable because $\dim H^1(X, \underline{\mathrm{Hom}}((\det E) \otimes L^{-1}, L)) = \dim H^1(X, \underline{(\det E)}^{-1} \otimes \underline{L}^2) = \dim H^0(X, \underline{(\det E)} \otimes \underline{L}^{-2}) = 0$ by the Serre duality and the fact that $\deg((\det E) \otimes L^{-2}) < 0$. Since E is indecomposable, the subbundle $[\phi]$ of E is degree 0, that is to say, $[\phi] \cong I$ and I is a maximal subbundle of E. Q.E.D.

The following lemma is a special case of the Atiyah's theorem ([2] Theorem 7).

Lemma 4.5. Let $\mathcal{E}(1, 1)$ be the set of the isomorphism classes of line bundle of degree 1. Then the map $\det : \mathcal{E}(2, 1) \longrightarrow \mathcal{E}(1, 1)$ is bijective.

Proof. Let $E \in \mathcal{E}(2, 1)$, then by Lemma 4.4 we have an exact sequence

$$0 \longrightarrow I \longrightarrow E \longrightarrow \det E \longrightarrow 0.$$

On the other hand, $\dim H^1(X, \underline{\mathrm{Hom}}(\det E, I)) = \dim H^1(X, \underline{(\det E)}^{-1}) = \dim H^0(X, \underline{\det E}) = 1$ by the Serre duality. Thus E is determined uniquely by $\det E$, that is to say, $\det : \mathcal{E}(2, 1) \longrightarrow \mathcal{E}(1, 1)$ is injective. Conversely, if $L \in \mathcal{E}(1, 1)$ is given, then there is a non-trivial extension E of L by I

$$0 \longrightarrow I \longrightarrow E \longrightarrow L \longrightarrow 0,$$

where $\det E = L$, because $\dim H^1(X, \underline{\mathrm{Hom}}(L, I)) = 1$. If $E \cong L_1 \oplus L_2$, then $I \not\subseteq L_1, L_2$ and $\dim H^0(X, \underline{\mathrm{Hom}}(I, L_i)) \geqq 1$, whence $\deg L_i > 0$ $(i = 1, 2)$. This is a contradiction since $\deg E = 1$. Thus E is indecomposable. Therefore, $\det : \mathcal{E}(2, 1) \longrightarrow \mathcal{E}(1, 1)$ is surjective.

Q.E.D.

Corollary 4.6. If $E \in \mathcal{E}(2, 1)$, then the set of the maximal subbundle of E is in bijective correspondence with the curve X.

Proof. X can be identified with the Jacobian variety of X. Let N be a maximal subbundle of E. Then N is of degree 0 by Lemma 4.4 and N naturally corresponds to a point of X. This map is injective by Lemma 1.5. Conversely, let N be the line bundle of degree 0 which corresponds to a point of X. By Lemma 4.5 we can find $E' \in \mathcal{E}(2, 1)$ such that $\det E = (\det E') \otimes N^2 = \det(E' \otimes N)$. Then, by the injectiveness of the map $\det : \mathcal{E}(2, 1) \longrightarrow \mathcal{E}(1, 1)$, we obtain $E = E' \otimes N$. Meanwhile, I is a maximal subbundle of E' by Lemma 4.4. Thus N is a maximal subbundle of E. This proves that the above map is surjective. Q.E.D.

Here we come to the Atiyah's theorem [1].

Theorem 4.7. The moduli space of isomorphism classes of P^1-bundles over X with $g = 1$ is the union of J_n (n runs over all negative integers), $\tilde{J}_0$ and two points P_0, P_1, where J_n and $\tilde{J}_0$ are the space of decomposable bundles and P_0 (or, P_1, respectively) is a class of indecomposable bundle with $N(P_0) = 0$ (or, $N(P_1) = 1$, resp.).

Proof. Decomposable cases are treated in Corollary 1.12. Next, we treat the indecomposable cases. First, note that $N(P(E)) \leqq 1$ and that $P(E)$ is decomposable if $N(P(E)) < 0$. In the case where $N(P(E)) = 0$, we have $\dim H^1(X, \mathcal{L}(P(E))^{-1}) = 1$ if and only if

$\mathfrak{d}(P(E)) = I$ and if $\mathfrak{d}(P(E)) \neq I$, we have $\dim H^1(X, \mathfrak{d}(P(E))^{-1}) = 0$ by the Serre duality. Thus, there is only one indecomposable bundle by Theorem 1.11 and it is P_0. Next, consider the case where $N(P(E)) = 1$. Let $E, E' \in \mathcal{E}(2, 1)$, then we have $\deg((\det E) \otimes (\det E')) = 0$. Therefore, we can find a line bundle N of degree 0 such that $N^2 = (\det E) \otimes (\det E')^{-1}$. Then, $\det E = \det (E' \otimes N)$ and by Lemma 4.5, $E = E' \otimes N$, that is, $P(E) = P(E')$. Hence, $P_1 = P(E)$ $(E \in \mathcal{E}(2, 1))$ is only one P^1-bundle with $N(P(E)) = 1$. Q.E.D.

The model of an element of J_n or $\widetilde{J}_0$ was given in Proposition 4.1. On the other hand, from Table I of Chapter III, Theorem 1.16, Lemma 1.15, Proposition 4.1 and Corollary 4.6, we get

Theorem 4.8.

(1) The model of P_0 is $\mathrm{elm}_{Q_1, Q_2} S_0$, where Q_1 is an ordinary point and Q_2 is an infinitely near point of Q_1 not on a base.

(2) The model of P_1 is $\mathrm{elm}_{R_1, R_2, R_3} S_0$, where R_1, R_2, R_3 are ordinary points on distinct bases and distinct fibres respectively.

(3) P_0 has only one minimal section and the self-intersection numbers of sections of P_0 are all even.

(4) The set of minimal sections of P_1 is in bijective correspondence with X and self-intersection numbers of sections of P_1 are all odd.

To consider the classification of biregular classes of ruled surfaces over X with $g = 1$ we prepare a lemma which was proved by T. Suwa [14] in the case where k is the complex number field.

Lemma 4.9. Let P_0 be a fixed point on X and $P(E)$ and $P(E')$ be decomposable P^1-bundles over X with $g = 1$ such that $N(P(E)) = N(P(E')) = 0$ and s (or, s', respectively) is a minimal section of $P(E)$ (or, $P(E')$, resp.). Then $P(E)$ is biregularly isomorphic to $P(E')$ if and only if there is an automorphism ϕ of X such that $\phi(P_0) = P_0$ and $\phi(P) = P'$ for $\pi(s \cdot s) = P_0 - P$, $\pi'(s' \cdot s') = P_0 - P'$.

Proof. First assume that ϕ stated above exists. Then $\phi^*(P(E'))$ has a minimal section s'' such that $\pi''(s'' \cdot s'') = P_0 - P$ where π'' is the projection of $\phi^*(P(E'))$. Thus $\phi^*(P(E')) \cong P(E)$ as P^1-bundle and so $P(E)$ is biregularly isomorphic to $P(E')$ by Theorem 0.3. Conversely, if Ψ is a biregular map of $P(E)$ to $P(E')$, then there is an automorphism ψ of X by Theorem 0.3 such that $\pi' \circ \Psi = \psi \circ \pi$. Thus if $\Psi(s) = s'$, $\pi'(\Psi(s) \cdot \Psi(s)) = \psi(\pi(s \cdot s)) = \psi(P_0) - \psi(P) \sim P_0 - P'$. Now, we shall identify X with its Jacobian variety and write the group operation of X multiplicatively. Let ϕ be the automorphism of X such that $\phi(Q) = P_0 \cdot \psi(Q) \cdot (\psi(P_0))^{-1}$. Then $\phi(P_0) = P_0$ and $\phi(P) = P'$ since $\psi(P_0) - \psi(P) \sim P_0 - P'$ and this ϕ is the required automorphism. Next, if $\Psi(s)$ is another minimal section (see Corollary 1.7), then $\psi(P_0) - \psi(P) \sim P' - P_0$ by Proposition 1.9 (iv). Let ϕ be the automorphism of X such that $\phi(Q) = P' \cdot (\psi(Q))^{-1} \cdot \psi(P)$ for all $Q \in X$. Then ϕ is the required automorphism. Q.E.D.

Thus we get

Theorem 4.10. The moduli space of biregular classes of ruled surfaces over X with $g = 1$ is the union of countable points P_n $(n \leqq 1)$ and $\tilde{X}$, where $\tilde{X}$ is the quatient space of X by automorphisms of X as an abelian variety.

Proof. Let $P(E)$ and $P(E')$ be P^1-bundles such that $N(P(E)) = N(P(E')) = n > 0$. Then there are $P, Q \in X$ such that $\mathfrak{d}(P(E)) = nP$ and $\mathfrak{d}(P(E')) = nQ$ as divisor classes. If ϕ is an automorphism of X such that $\phi(P) = Q$, then $\phi^*(P(E')) \cong P(E)$ as P^1-bundle since $\mathfrak{d}(\phi^*(P(E'))) = nP$. Thus $P(E)$ is biregularly isomorphic to $P(E')$ by Theorem 0.3 and the class of this $P(E)$ is P_n. P_0, P_1 are the same as in Theorem 4.7. It is clear by Lemma 4.9 that the space of biregular classes of decomposable ruled surfaces (i.e. decomposable as P^1-bundle) with $N(P(E)) = 0$ is $\tilde{X}$. Since P_0 has only one minimal section and every element of $\tilde{J}_0$ has two minimal sections, P_0 is never contained in $\tilde{X}$. Q.E.D.

(C). The case where $g = 2$.

Theorem 4.11. The moduli space of $\mathfrak{P}^-_X$ over X with $g = 2$ is the union of J_n (n ranges over all negative integers), $\tilde{J}_0$, P_{-2}, X

and $\hat{J}_0$ where J_n and $\tilde{J}_0$ are the same as in Corollary 1.12, P_{-2} is a point and where $\hat{J}_0$ is the variety which is obtained by the dilatation at the unity from the Jacobian variety of X. P_{-2}, X and $\hat{J}_0$ correspond to the indecomposable bundles of $N(P(E)) = -2, -1,$ and 0 respectively.

Proof. It is sufficient if one proves the assertion in the indecomposable case. Thus we need consider only the case $0 \geqq N(P(E)) \geqq -2$. If $N(P(E)) = -2$, then $\dim H^1(X, \mathfrak{d}(P(E))^{-1}) \leqq 1$ and the equality is fulfiled if and only if $\mathfrak{d}(P(E))^{-1}$ is the canonical divisor class by the Serre duality. Hence an indecomposable bundle $P(E)$ with $N(P(E)) = -2$ is unique by Theorem 1.11 and it is P_{-2}. If $N(P(E)) = -1$, then we obtain $\dim H^1(X, \mathfrak{d}(P(E))^{-1}) = \dim H^0(X, \mathfrak{d}(P(E))^{-1})$ by the Riemann-Roch theorem. Thus, if $\dim H^1(X, \mathfrak{d}(P(E))^{-1}) \geqq 1$, then the divisor class defined by $\mathfrak{d}(P(E))^{-1}$ contains a divisor P, where P is a point of X. Here, the inequality $\dim H^1(X, \mathfrak{d}(P(E))^{-1}) > 1$ does not hold. For, if so, there are two points P, Q of X such that P and Q are linearly equivalent as one point divisors. However, this fact happens only in the case where X is rational. Hence, $\dim H^1(X, \mathfrak{d}(P(E))^{-1}) = 1$. This latter equality holds, if and only if $|\mathfrak{d}(P(E))^{-1}| = P$, for one point divisor P. If $\dim H^1(X, \mathfrak{d}(P(E))^{-1}) = 0$, we see easily that $P(E)$ is decomposable (Theorem 1.11). Therefore, there is a bijective correspondence between

X and the set of indecomposable P^1-bundles with $N(P(E)) = -1$. Finally, consider the case where $N(P(E)) = 0$. Then $\dim H^1(X, \mathfrak{d}(P(E))^{-1}) = 1$ if $\mathfrak{d}(P(E)) \neq I$, and $\dim H^1(X, \mathfrak{d}(P(E))^{-1}) = 2$ if $\mathfrak{d}(P(E)) = I$. Since the set $\{\mathfrak{d}(P(E)) \mid N(P(E)) = 0\}$ is bijective with J (= the Jacobian variety), the set of indecomposable bundles with $N(P(E)) = 0$ is $\hat{J}_0$ by Theorem 1.11.

Combining Theorem 1.16, Theorem 4.11, Lemma 4.2 and Table II of Chapter III, the models of indecomposable bundles of $\mathfrak{P}_X^-$ over X with $g = 2$ are the following :

(1) The domel of P_{-2} is $\mathrm{elm}_{P_1, P_2, Q, R}\, S_0$, where $P_1 + P_2$ is a canonical divisor of a base, Q is an arbitrary point on the same base and R is an infinitely near point of Q not on a base.

(2) The model of an element of X is $\mathrm{elm}_{P, Q, R}\, S_0$, where P is the point on a base corresponding to the given point of X, Q is a point on the same base such that $\dim | P + Q | = 0$ and where R is an infinitely near point of Q not on a base.

(3) The models of $\hat{J}_0$ are the following : Let D_0 and D_∞ be positive divisors on X such that $D_0 - D_\infty = \mathfrak{d}(P(E))$ and that $\deg D_0$ is minimal.

(i) If $\mathfrak{d}(P(E)) = I$, then $P(E) = \mathrm{elm}_{P, Q}\, S_0$, where P is an ordinary point and Q is an infinitely near point of Q not on a base.

(ii) If $D_\infty = P_1$ and $D_0 = P_2$, then $P(E) = \mathrm{elm}_{P_1', P_2', Q, R}\, S_0$, where P_1' (or, P_2', respectively) is the point lying on $(0) \times X$ (or, $(\infty) \times X$, resp.) such that $\pi(P_1') = P_1$ (or, $\pi(P_2') = P_2$, resp.), Q is a point on $(0) \times X$ such that $\dim | P_1' + Q | = 0$ and R is an infinitely near point of Q not on a base.

(iii) If $D_\infty = P_1 + P_2$ and $D_0 = Q_1 + Q_2$, then $P(E) = \mathrm{elm}_{P_1', P_2', Q_1', Q_2'} S_0$, where P_1' and P_2' are the points lying on the same base such that $\pi(P_1') = P_1$ and $\pi(P_2') = P_2$ and where Q_1' and Q_2' are the points such that $\pi(Q_1') = Q_1$ and $\pi(Q_2') = Q_2$ and that P_1', Q_1', Q_2' lie on distinct bases respectively.

The following theorem is a special cases of Theorem 3.17.

Theorem 4.13. Let the genus of X be 2. Then there are P^n-bundle spaces V_n $(n = 1, 2)$ over J_n and we have the following :

(1) There is a surjective map $f_1 : V_1 \longrightarrow \mathcal{P}_X^1$ and $\# f_1^{-1}(P(E)) = \# \mathcal{A}(P(E))$ is at most four for all $P(E) \in \mathcal{P}_X^1$.

(2) There is a surjective map f_2 of a Zariski open set of V_2 to $\mathcal{P}_X^2$ and $\dim f_2^{-1}(P(E)) = 1$ for all $P(E) \in \mathcal{P}_X^2$.

(D) The case where $g = 3$

Taking Theorem 3.8 and Theorem 3.17 into account, we have

Theorem 4.14. Let the genus of X be 3. Then there are P^{n+1}-bundle space V_n $(n = 1, 2, 3)$ over J_n and we have the following :

(1) There is a surjective map f_1 of V_1 to $\mathcal{P}_X^1$ and $\# f_1^{-1}(P(E)) = \# \mathcal{A}(P(E))$ is at most six for all $P(E) \in \mathcal{P}_X^1$.

(2) There is a surjective map f_2 of a Zariski open set of V_2 to $\mathcal{P}_X^2$ and $\# f_2^{-1}(P(E)) = \# \mathcal{A}(P(E))$ is finite for all $P(E) \in \mathcal{P}_X^2$.

(3) There is a surjective map f_3 of a dense set of V_3 which contains a Zariski open set of V_3 to $\mathcal{P}_X^3$ and $\dim f_3^{-1}(P(E)) = 1$ for all $P(E) \in \mathcal{P}_X^3$.

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